Springer Monographs in Mathematics

Rafael López

Constant Mean Curvature Surfaces with Boundary

Springer

S*pringer* **M***onographs* in **M***athematics*

For further volumes:
www.springer.com/series/3733

Rafael López

Constant Mean Curvature Surfaces with Boundary

Springer

Rafael López
Department of Geometry and Topology
University of Granada
Granada, Spain

ISSN 1439-7382 ISSN 2196-9922 (electronic)
Springer Monographs in Mathematics
ISBN 978-3-642-39625-0 ISBN 978-3-642-39626-7 (eBook)
DOI 10.1007/978-3-642-39626-7
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013948448

Mathematics Subject Classification (2010): 53, 53A, 53A10, 53C42, 35J, 35J60

© Springer-Verlag Berlin Heidelberg 2013
This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.
The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.
While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Dedicated to Elia and Irene, my daughters

Preface

Surfaces of constant mean curvature (*cmc surfaces*) play an important role in classical differential geometry because they are the local solutions of one of the oldest problems in geometry, namely, the isoperimetric problem: find those surfaces of least area enclosing a prescribed volume. These surfaces are of great interest not only for mathematicians, but also for physicists and engineers since they provide mathematical models in contexts where a physical system seeks a state of least energy. A number of texts are available which deal with this type of surface in more or less depth. The classical book of H. Hopf [Hop83] contains interesting material that can be used as an introduction. We also refer to Nitsche and Osserman for minimal surfaces [Nit89, Oss86] and the recent book of K. Kenmotsu [Ken03]. See also [Boy59, EJ07, Law80, Opr00].

This book introduces the reader to that part of the subject concerning 'surfaces with boundary'. Our aim is to present as much as possible of the theory of compact constant mean curvature surfaces spanning a given boundary curve, selecting the results that represent the core of this theory. The initial motivating problem is the simplest case of boundary, say a circle, and we ask what type of symmetries of the circle are inherited by the surface that spans it. We describe various methods, such as, the tangency principle, the Alexandrov method, the flux formula and the Dirichlet problem and we shall emphasize the geometric aspects of these techniques in the context of the theory of surfaces with constant mean curvature. Most of the results are given in Euclidean space and in particular in the three-dimensional case, but we can consider other ambient spaces and dimensions. In this monograph we only extend the discussion to hyperbolic space and we dedicate the last chapter to the study of these surfaces in Lorentz-Minkowski space. The references that appear in this book are only a small part of the bibliography in this theory.

One of the main goals of this book is to help graduate students to get started on research in the theory of surfaces in Euclidean space with constant mean curvature. We have tried to write a reasonably self-contained text so that the reader can learn the techniques of the subject in such a way that they will begin with basic methods and results and end with recent research topics. We do assume that the reader knows the elementary theory of surfaces, as presented for example in

do Carmo [Car76] and Montiel and Ros [MR09] and the basic theory of integration and submanifolds in Riemannian manifolds as described in selected parts of [Car92, Chv94].

The text is organized as follows. In Chap. 1 we present an historical review of surfaces with constant mean curvature motivating their study by the classical isoperimetric problem. Historically there are two parallel problems depending on whether we consider only one constraints (the volume) or two constraints (the volume and the boundary). In the first case, we recall the different characterizations of the sphere within the family of closed surfaces with constant mean curvature. The absence of new examples brought as a consequence a great effort to find new surfaces (and techniques), finally succeeding in the discovery by H.C. Wente of a torus immersed in Euclidean space with constant mean curvature. This opened new avenues of research in the theory. For surfaces with boundary, Douglas and Radó considered the Plateau problem for minimal surfaces and later, in the 1950s, Heinz and others considered the general case of mean curvature.

In Chap. 2, anticipating the main discussions of the book, we focus on the basic properties of cmc surfaces. We derive the first variation of the area and we give the variational characterization of a cmc surface. We give the notion of stability of a surface when we compute the second variation of the area. Next we prove the Hopf theorem, introducing complex coordinates on a cmc surface. Following this, we compute the Laplacian of the position vector and the Gauss map. With these techniques we obtain height estimates for a cmc graph and finally we prove the classical results of Jellet and Barbosa-do Carmo that characterize the sphere within the family of closed cmc surfaces under the corresponding hypothesis that the surface is starshaped and stable, respectively.

Chapter 3 introduces the comparison principle and the tangency principle, two of the main tools employed in this book. These are obtained first by writing a surface locally as the graph of a function $z = u(x, y)$. Then the constancy of the mean curvature implies that u satisfies a second order partial differential equation of elliptic type. The ellipticity of the equation allows the use of the classical Hopf maximum principle. We will obtain an initial set of results on compact cmc surfaces with planar boundary giving conditions which ensure that the surface lies on one side of the boundary plane. Comparing the surface with spheres and cylinders, we obtain characterizations of a sphere if the surface is included in the closure of a Euclidean ball or a cylinder whose radii are related to the value of the mean curvature of the surface.

In Chap. 4 we study embedded cmc surfaces using the reflection technique of Alexandrov. With this method we will give conditions determining whether the symmetries of the boundary are inherited by the surface, in particular, when the boundary is a plane curve. We employ the method of reflection by means of planes orthogonal to the plane containing the boundary, proving that, under some conditions, if the boundary is a circle the surface is rotational. In this context, we also use the reflection method with planes parallel to the boundary plane, obtaining results that prove that the surface is a graph. Finally, we use the Alexandrov method for embedded cmc surfaces whose boundary lies in a sphere and we give conditions that ensure that the surface lies on one side of the sphere.

In Chap. 5 we study how the geometry of a given closed curve imposes restrictions on the existence of surfaces with constant mean curvature that span such a curve. We shall obtain a flux formula that will play an important role, not only in this chapter but throughout the rest of the book. With the aid of the flux formula, and jointly the tangency principle, we will derive some geometric configurations where the surface is contained in a Euclidean ball or a cylinder whose size is related with the value of mean curvature of the surface. In the last section, we will prove that if an embedded surface with convex boundary is transverse to the boundary plane then it lies on one side of this plane.

Chapter 6 addresses the study of the area and the volume of a compact cmc surface with boundary. The control of the area or the volume will give information on the geometry of the surface. The chapter begins by obtaining a monotonicity formula for the area in terms of the height of the surface and we will show that the solutions of the isoperimetric problem for convex planar domains are graphs if the volume is sufficiently small. In the last section, we will consider embedded surfaces with large volume and we will prove that, under some conditions, the surface converges to a large spherical cap if the volume is sufficiently big.

Chapter 7 treats the case where the given boundary is a circle. When the topology of the surface is a disk, we shall characterize a spherical cap under hypotheses on the area. In this chapter, it will be shown that the planar disk and the spherical cap are the only stable disks with constant mean curvature spanning a circle.

Chapters 8 and 9 are devoted to the Dirichlet problem of the constant mean curvature equation. First we consider the case where the domain is bounded, whereas in Chap. 9 the domain is unbounded. We will describe the techniques used to solve the Dirichlet problem, namely, the continuity method and the Perron method. We shall give a set of existence results when the domain is convex under hypotheses on the curvature and length of the boundary curve or the area of the domain. If the domain is not convex, we assume a uniform circle exterior condition. In order to get the desired estimates, we shall employ pieces of rotational cmc surfaces as barriers. In addition, we will give estimates of the height of a cmc graph. If the boundary values are bounded, the estimates depend only on the mean curvature and the boundary data. When the domain is bounded, we also obtain estimates of the graph in terms of the area of the domain.

Chapter 10 concerns the study of compact cmc surfaces in hyperbolic space $\mathbb{H}^3$ and we ask questions similar to those posed in Euclidean space. A basic difference in this ambient space is the variety of umbilical surfaces because, besides totally geodesic surfaces and spheres, there are equidistant surfaces and horospheres. We shall see that the problems differ depending on the value of the mean curvature H. For example, if $|H| > 1$, the surface exhibits behaviour similar to that in $\mathbb{R}^3$, if $|H| = 1$, the surface has some similarities with a minimal surface of $\mathbb{R}^3$ and finally, if $|H| < 1$, the nature of the problem has no equivalent in Euclidean space.

Chapter 11 deals with the Dirichlet problem for geodesic graphs in the hyperbolic setting. We shall consider the solvability of the problem when the domain is included in a geodesic plane, an equidistant surface and a horosphere. Here we restrict to the case that the domain is bounded and strictly convex.

Finally, in Chap. 12 we consider cmc surfaces with boundary in Lorentz-Minkowski space $\mathbb{L}^3$. Here we study spacelike surfaces because the induced metric on the surface is Riemannian. We shall characterize the umbilical surfaces as the only spacelike compact surfaces of constant mean curvature spanning a circle. We also treat techniques such as the tangency principle and the flux formula, and finally, we shall consider the Dirichlet problem for the constant mean curvature equation.

There is an appendix, where we derive the first and second variation of the area, and we finish with an open problems section.

We point out some topics on cmc surfaces that are beyond the scope of this book.

1. Complete surfaces with constant mean curvature. After the discovery by H.C. Wente of an immersed torus with constant mean curvature in Euclidean space [Wen86], a great deal of work has been focused on the description of complete and closed cmc surfaces with arbitrary topology. This effort was initiated by N. Kapouleas, who constructed surfaces with high genus. See [Kap90, Kap91, Kap92].
2. Existence of parametric constant mean curvature surfaces. The techniques are based on functional analysis. The pioneering results were established by, among others, E. Heinz and S. Hildebrandt. The two texts of Struwe are good introductions to the topic [Str88, Str00].
3. Existence of non-parametric surfaces whose mean curvature is not constant. The constant mean curvature equation is just one of a whole family of equations of divergence type, which have been widely studied in the theory of partial differential equations. The text of Gilbarg and Trudinger [GT01] is a suitable starting point for the techniques and the first results.
4. The DPW method. For a non-minimal cmc surface, J. Dorfmeister, F. Pedit and H. Wu developed a generalized Weierstrass representation in terms of holomorphic functions by combining the Sym-Bobenko formula with integrable systems methods [DPW98]. Using loop groups, this technique can be used to construct cmc surfaces. We refer to [FKR05, Hel01].
5. Capillarity. The condition that the boundary curve is prescribed is replaced by the fact that the angle between the surface with a given support is constant. We refer to [Fin86, Lan02].
6. Minimal surfaces, that is, surfaces with zero mean curvature. See [BaCo86, CM11, DHKW92, Nit89, Oss86].
7. Generalizations in other ambient spaces, such as homogeneous spaces. In the last decade, there has been an intensive effort to develop the theory of cmc surfaces, including minimal surfaces, in Thurston's eight models for 3-dimensional geometries [Thu97]. This interest is due after the recent work of U. Abresch and H. Rosenberg concerning the extension of the Hopf theorem [AR04]. See the surveys [DHM09, MP12]. Some works that consider the topics of this book in these spaces are [Bar13, CS12, Daj06, ET08, ET11, EFR10, FM11, FR12, HRS09, Lop12, NEST08, Pin09, Sen11, Spr07].

This monograph is an outgrowth of a series of seminars entitled "Surfaces with constant mean curvature in Euclidean space" that I gave in 2010 at the University

of Granada. The origin of some of the material can also be found in the author's articles [Lop96, Lop05, Lop06a, Lop10a].

Some of the results of this book which relate to my own research grew out of discussions with colleagues and collaborators. Firstly, I am grateful to Sebastián Montiel, the advisor of my Ph. Doctoral Thesis [Lop96], who first introduced me to the field of differential geometry and, in particular, to the topic of surfaces with constant mean curvature. Also, I am indebted to Antonio Ros who has always stimulated my work and with whom I have enjoyed many fruitful discussions. Finally, I wish to thank Luis J. Alías, Bennett Palmer, Joaquín Pérez and Miguel Sánchez for valuable conversations.

Granada, Spain Rafael López
May 2013

Contents

1 A Brief Historical Introduction and Motivations 1

2 Surfaces with Constant Mean Curvature 13
2.1 The Variational Formula for the Area 13
2.2 The Hopf Differential 24
2.3 Elliptic Equations for cmc Surfaces 26

3 The Comparison and Tangency Principles 37
3.1 The Comparison Principle 37
3.2 The Tangency Principle 44
3.3 Applications of the Tangency Principle 49

4 Constant Mean Curvature Embedded Surfaces 55
4.1 The Alexandrov Reflection Method 56
4.2 The Alexandrov Method with Horizontal Planes 68
4.3 Embedded Surfaces with Boundary in a Sphere 72

5 The Flux Formula for Constant Mean Curvature Surfaces 81
5.1 The Flux Formula 81
5.2 Uniqueness on Right Cylinders 89
5.3 Small Surfaces with Constant Mean Curvature 92
5.4 The Brito, Earp, Meeks and Rosenberg Theorem 97

6 The Area and the Volume of a Constant Mean Curvature Surface . 103
6.1 A Monotonicity Formula for the Area 103
6.2 Consequences of the Height Estimate 108
6.3 Surfaces with Small Volume 110
6.4 Surfaces with Large Volume 115

7 Constant Mean Curvature Disks with Circular Boundary 123
7.1 Disks with Small Area 123
7.2 Stable cmc Disks 127
7.3 Other Area Estimates of a cmc Disk 131

8 The Dirichlet Problem of the cmc Equation 133
8.1 The Method of Continuity 134
8.2 The Bounded Slope Condition 142
8.3 Graphs on Convex Domains 145
8.4 The Exterior Circle Condition 156
8.5 Radial Graphs in Euclidean Space 162

9 The Dirichlet Problem in Unbounded Domains 173
9.1 Height Estimates 173
9.2 Graphs on Convex Domains 179
9.3 The Exterior Circle Condition 182

10 Constant Mean Curvature Surfaces in Hyperbolic Space 189
10.1 Basics on Hyperbolic Geometry 190
10.2 The Flux Formula 197
10.3 Embedded cmc Surfaces in $\mathbb{H}^3$ 200
10.4 Graphs in Hyperbolic Space 205
10.5 The Case $|H| \leq 1$ 211
10.6 Surfaces with Circular Boundary 214

11 The Dirichlet Problem in Hyperbolic Space 219
11.1 Preliminaries and Definitions 219
11.2 Graphs on a Horosphere 223
11.3 Graphs on a Geodesic Plane and an Equidistant Surface 228

12 Constant Mean Curvature Spacelike Surfaces in Lorentz-Minkowski Space 235
12.1 Basics on Lorentz-Minkowski Space 236
12.2 The Mean Curvature of a Spacelike Surface 240
12.3 Spacelike Surfaces with Non-empty Boundary 243
12.4 The Flux Formula 246
12.5 A Monotonicity Formula for the Area 251
12.6 The Dirichlet Problem 253

Appendix A The Variation Formula for the Area and for the Volume . 261

Appendix B Open Questions 269

References 273

Index 289

Chapter 1
A Brief Historical Introduction and Motivations

The problems that we consider in this book are easily explained to a non-mathematician and can be introduced as follows. Let us take a drinking straw with circular cross-section and lower one of its ends into a container filled with a soap solution. When we extract the straw, a soap film has formed which coincides with the flat disk whose boundary is the circular rim at the straw's end. Now we slowly pump air into the straw from the other end and we see that the flat disk changes into a soap bubble attached to the border. On the end into which we are pumping air, we place a finger so that the air cannot escape from the straw. The soap bubbles we always observe are spherical caps. Once the spherical cap is formed, we can perform small movements of the bubble without removing our finger and without destroying the soap bubble in such way that the bubble remains attached to the straw. The bubble will perturb and change shape, but when we stop and the soap bubble attains a new equilibrium, the surface formed is the initial spherical cap again. This cap has the property that its *mean curvature is constant* on the surface. In our experiment, this constant is proportional to the difference in air pressure on both sides of the soap film according to the Laplace equation. For example, in the flat disk, the first formed soap film, the pressures on each side of the surface are the same and agree with the atmospheric pressure. In this case, the mean curvature of the surface is zero. Once we introduce air into the interior of the straw and the spherical cap has been formed, the pressure of the air inside the straw is greater than the atmospheric pressure on the other side of the soap bubble.

From a physical viewpoint, the surface has been constructed under two constraints. First, the boundary, which is fixed and, in our case, is the circular rim of the straw. Second, the volume of air enclosed by the bubble's surface together with the straw which also remains fixed. In this sense, a spherical cap is a solution of a variational problem because when we perturb the surface preserving both constraints, the energy of the soap bubble is bigger since it increases as the surface area grows. Then physical equilibrium is attained again when the surface minimizes its area, and this occurs when this surface is spherical. From the mathematical viewpoint, it is natural to ask:

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_1,
© Springer-Verlag Berlin Heidelberg 2013

What is the shape of a *mathematical* soap bubble with circular boundary?

More generally, we want to investigate the relationship between the shape of the rim and that of the soap bubble formed. The contribution of this book is to explore the answers to the following question:

Given a closed curve Γ, what is the shape of a compact surface of constant mean curvature whose boundary is Γ?

Returning to the experiment, if we continue to blow air into the straw, the bubble eventually becomes detached from the straw, taking the shape of a sphere; the so-called soap bubble that any child makes with a bubble wand. Again the soap bubble minimizes area but now there is only one constraint, namely, the volume enclosed by the bubble. In this case, and from a topological viewpoint, the soap bubble is a closed surface, that is, a compact surface without boundary. Closed surfaces of constant mean curvature have been studied from the beginning because the sphere was the first example of such a surface.

In differential geometry, the mean curvature of a surface appears when we consider the *classical isoperimetric problem*, which can be stated as follows:

Among all surfaces that enclose a given amount of volume, find the surface of least area.

The answer is the round sphere, as was shown by H.A. Schwarz using previous ideas of J. Steiner [Swz84, Stn42]. If we only look for infinitesimal solutions of the isoperimetric problem, the area of a compact surface immersed in Euclidean space is stationary with respect to all volume preserving variations if and only if the immersion has constant mean curvature. In other words, surfaces with constant mean curvature are the critical points of the area functional with respect to local deformations that preserve the enclosed volume. In the case of zero mean curvature at each point, the surface is called a *minimal surface*. The study of surfaces with constant mean curvature is of interest in itself, besides the isoperimetric problem, and it is a common thread in many works in differential geometry. As N. Hitchin asserted in his contribution to the volume "Mathematics Unlimited—2001 and Beyond", referring to the surfaces of constant mean curvature [Hit00, p. 577]:

> "[...] the objects, though perhaps not the methods, would have been recognizable by participants at an International Congress a hundred years ago."

Surfaces of constant mean curvature, or simply cmc surfaces, are used as mathematical models in physical settings where the energy is proportional to the surface

area. For example, they are models of the rise of a liquid in a thin tube introduced into a tank of liquid in the absence of gravity that ascends or descends by the capillary action. The first works on capillarity were initiated by T. Young and P.S. Laplace at the beginning of the 19th century [Lap05, You05]. The notion of mean curvature appeared around 1810 in elasticity problems studied by S. Germain who coined the name 'mean curvature' [Ger31], although it was J.B. Meusnier who first considered the sum of the principal curvatures to characterize surfaces that locally minimize area [Meu85].

Other examples in physics where cmc surfaces arise are the so-called *interfaces*. An interface is the region where two different homogeneous phases meet, such as two immiscible liquids or between a liquid and air. We may consider this region separating both media to have negligible thickness in such way that the interface is modeled by a mathematical surface. In the absence of gravity, the shape of this interface changes until it attains a state of physical equilibrium, which occurs when the interface minimizes the energy. In this setting, the energy of the interface is proportional to its surface area. Then the pressure difference $P_e - P_i$ between the exterior and the interior across the interface S is proportional to the surface tension γ of S and the shape of the interface is governed by the *Laplace equation*

$$P_e - P_i = 2H\gamma. \tag{1.1}$$

Here H denotes the mean curvature of the surface S and γ is a constant depending only on the materials. In the case where the pressures are constant, the mean curvature of S is constant. Soap bubbles and soap films are examples of interfaces in this context. If we consider liquid drops resting (or hanging) on a plane in the presence of gravity, the Laplace equation (1.1) is modified, adding an extra term that reflects the weight of the drop. In this setting the mean curvature of the liquid-air interface is a linear function of the vertical coordinate, say, towards where the gravity acts. The literature on interfaces and capillarity is extensive and we do not include here a complete list. From a physical viewpoint, we refer to Adamson and Gast [AG82], Boys [Boy59] and the most recent texts of de Gennes, Brochard-Wyart and Quéré [GBQ04] and Langbein [Lan02]. In mathematics, we cite Finn [Fin86] and Isenberg [Ise92].

A special topic is the case of minimal surfaces. By the Laplace equation (1.1), the mean curvature is zero on the interface if the pressures on both sides are equal. Mathematically this means that a minimal surface is a critical point of the area for arbitrary variations without the need to preserve the volume. Although the equation $H = 0$ is the simplest one, the theory is extremely rich. Here we mention the classical texts of Nitsche [Nit89] and Osserman [Oss86] and, recently, the book of Colding and Minicozzi [CM11]. See also [BaCo86, CM99, DHKW92, FT91, Hof05, Law80] and references therein.

As we have pointed out, motivated by the round sphere, it is natural to consider cmc closed surfaces as a first step. For many years, the only known example of a closed cmc surface was the sphere and the works of geometers focused on characterizing it. Already by 1853, J.H. Jellet had proved that the sphere is the only star-

shaped surface of constant mean curvature [Jel53]. In 1900 H. Liebmann showed that an ovaloid of constant mean curvature must be a sphere [Lie00].

Half a century later, two results were proved which are milestones in the theory. In 1951, H. Hopf [Hop83] proved, associating to each cmc surface a holomorphic differential 2-form, that

> The sphere is the only closed cmc surface of genus 0.

Later, in a sequence of articles between 1956 and 1962, A. D. Alexandrov [Ale56, Ale62] showed that

> The sphere is the only closed embedded cmc surface.

This result was indeed surprising not for the statement, which was expected, but for the method that Alexandrov used to prove it, which was completely different to that of Liebmann and Hopf. The technique, called the *reflection method of Alexandrov*, is based on a clever use of the maximum principle for linear elliptic equations and it has been utilized by a number of authors in differential geometry and in the theory of partial differential equations. Combining these two results, a last theorem due to J.L. Barbosa and M. do Carmo completes this scenario [BC84]:

> The sphere is the only closed stable cmc surface.

Until the eighties, the sphere was the only known closed cmc surface in Euclidean space and many geometers asked if it was the only example. This is the so-called Hopf conjecture, or 'Hopf's problem' of M. Spivak in [Spi79, vol. IV, p. 451], although Hopf's quote is [Hop83, p. 131]:

> "The question whether there exist closed surfaces of genus ≥ 1 with $H = c$ and with self-intersections (i.e., not simple) which are not spheres remains unanswered."

Finally, another milestone in the theory, H. C. Wente demonstrated in 1986 the existence of an immersed torus in $\mathbb{R}^3$ with constant mean curvature [Wen86], introducing an exciting development of finding closed cmc surfaces with arbitrary genus, that continues today.

After these classical results, the study of cmc surfaces without boundary in Euclidean space and, in general, of hypersurfaces of this type in $\mathbb{R}^n$, has followed different paths. One of them was the discovery by R.C. Reilly in 1978 of a new proof of the Alexandrov theorem, more analytic but less geometric [Rei77]. Next, S. Montiel and A. Ros explored other methods obtaining results of Alexandrov type for more general ambient spaces and curvatures [MR91, Ros87, Ros88]. Another approach

was the one initiated by H.C. Wente with his discovery. The technique he developed established a relation between cmc surfaces and certain integrable systems that allowed a deep study of cmc tori in $\mathbb{R}^3$ [Abr87, Bob91, PS89, UY92, Wal89].

New examples of soap bubbles were produced by N. Kapouleas who constructed in 1991–1992 closed cmc surfaces immersed in $\mathbb{R}^3$ with arbitrary genus [Kap91, Kap92]. Motivated by the work of R. Schoen on the construction of complete, constant scalar curvature metrics [Sch88], the technique is based on a perturbing argument from an initial configuration in equilibrium by gluing Delaunay surfaces. Along the lines of contact, the surface is not smooth, but an analysis of the kernel of the Jacobi operator and the Leray-Schauder fixed-point theorem allows slight perturbations of the initial surface obtaining a smooth surface with constant mean curvature. The idea behind his construction method was used later by authors searching for compact cmc surfaces: see for example [JP05, MPP01, MPP05, Rat01].

With respect to complete non-compact surfaces, the classical examples, discovered by C.E. Delaunay in 1838, are the (non-spherical) cmc surfaces of revolution: cylinders, unduloids, nodoids and catenoids, whose generating curves are the traces of the focus of a conic (circle, ellipse, hyperbola and parabola) that rolls on a straight line which is the axis of revolution of the surface [Del41]. Besides these surfaces, N. Kapouleas constructed examples of complete cmc surfaces with finite topology and more complicated than a cylinder [Kap90]. See also [Jle09, MP01, MP02]

W.H. Meeks III showed in 1998 that the ends of a complete cmc surface properly embedded in $\mathbb{R}^3$ must be cylindrically bounded, as is the case for the Delaunay surfaces. As a consequence, it was shown that there do not exist complete cmc surfaces properly embedded in $\mathbb{R}^3$ with only one end [Mee88]. Later, in 1989, N. Korevaar, R. Kusner and B. Solomon proved that the only properly embedded complete cmc surfaces with only two ends are the Delaunay surfaces [KKS89]. More results on complete cmc surfaces appear in [BK12, GK99, GKS00, GKS03, GKS07, KK93, KKR06, KMP96, PR10].

Recently, another method has appeared for constructing cmc surfaces by using loop groups. This is the *DPW method* initiated by J. Dorfmeister, F. Pedit and H.Y. Wu [DPW98]. This technique is motivated by the fact that away from umbilical points, a cmc surface is described in terms of a sinh-Gordon equation where, in order to be integrated, the theory of integrable systems can be applied. See [DH98, DH00, DH03, DW93, DW08, KKRS05, KMS00, KSS04] and the monographs [FKR05, Hel01].

There are several web sites containing 'galleries' of cmc surfaces. Their addresses are listed below (accessed in May, 2013):

1. The Center for Geometry, Analysis, Numerics and Graphics (GANG) at the University of Massachusetts, Amherst:

   `http://www.gang.umass.edu/`

2. Indiana University, maintained by Mattias Weber:

   `http://www.indiana.edu/~minimal/`

3. The University of Granada, maintained by Francisco Martín:

   http://www.ugr.es/~fmartin

4. The Scientific Graphics Project at the Mathematical Sciences Research Institute:

   https://secure.msri.org/about/sgp/jim/geom/index.html

5. The University of Tubingen, maintained by Nicholas Schmitt:

   http://www.mathematik.uni-tuebingen.de/ab/Differentialgeometrie/gallery/

Some of the above problems can be posed in other ambient spaces and dimensions. Among the variety of cases, there is no doubt that one of the most interesting is the study of compact cmc surfaces in the three-dimensional sphere $\mathbb{S}^3$. From the beginning, interest was focused on the family of compact minimal surfaces. In contrast to the Euclidean space, there are compact minimal surfaces in $\mathbb{S}^3$, as for example, an equator $\mathbb{S}^2 \times \{0\}$ of $\mathbb{S}^3$. A result of F.J. Algrem proves that the equator is the only immersed compact minimal surface in $\mathbb{S}^3$ with genus 0, obtaining the analogous result of Hopf in the setting of $\mathbb{S}^3$. In fact, he used Hopf differentials in the same way as they are used in the Euclidean case. See also [Law70a].

For the next genus g, $g = 1$, the Clifford torus defined by the product of two circles of radius $1/\sqrt{2}$, namely,

$$\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) = \left\{(x, y, z, t) \in \mathbb{S}^3 : x^2 + y^2 = z^2 + t^2 = \frac{1}{2}\right\},$$

is a minimal torus in $\mathbb{S}^3$, which is also embedded and rotationally symmetric. If we increase the genus g, H.B. Lawson proved the existence of a compact embedded minimal surface in $\mathbb{S}^3$ for each $g \geq 2$ [Law70a]. Later, some techniques have been developed to construct new compact minimal surfaces for $g \geq 2$ [Kap11, KY10, KPS88]. In [KY10], N. Kapouleas and S.D. Yang employed Clifford tori connected by catenoid necks imitating the way as Kapouleas had used Delaunay surfaces to find compact cmc surfaces in Euclidean space [Kap91].

However, the case of genus 1 presented an interesting attraction for geometers due to the lack examples. In 1970, H.B. Lawson conjectured that the Clifford torus was the only compact embedded minimal surface in $\mathbb{S}^3$ of genus 1 [Law70b]. Many efforts were made to solve this problem and some partial affirmative answers were obtained [MN13, MR85, Ros95, Urb90]. The difficulty of the problem is highlighted by the fact that the Lawson's conjecture was included in the Millennium Problems series [Nor05]. Recently S. Brendle has proved affirmatively Lawson's conjecture [Bre13]. We remark that the embeddedness assumption is necessary because H.B. Lawson constructed an infinite family of minimal immersions from a torus into $\mathbb{S}^3$ [Law70a]. If the mean curvature is a non-zero constant, U. Pinkall and I. Sterling conjectured that all embedded cmc tori in $\mathbb{S}^3$ are surfaces of revolution [PS89]. This fact has been confirmed by B. Andrews and H. Li [AL12], giving the

classification of rotationally symmetric cmc surfaces of $\mathbb{S}^3$ and completing the work initiated by O. Perdomo in [Per10].

Running alongside this flow of results on complete and closed cmc surfaces, there is another trend of research corresponding to surfaces with boundary. Roughly speaking, the question is the following:

> Given a closed curve $\Gamma \subset \mathbb{R}^3$ and $H \in \mathbb{R}$, find the surface with constant mean curvature H spanning Γ.

An approximation to this question is again by means of the isoperimetric problem. Then we need that the surface is compact in order to consider its area. Now the surface has non-empty boundary and there are two versions of the problem depending on whether or not we consider the volume as a constraint of the problem.

> Given a closed curve $\Gamma \subset \mathbb{R}^3$, find the surface of least area with boundary Γ.

The surfaces that are solutions of this problem are minimal surfaces. If we prescribe a volume enclosed by the surface, the problem is the following:

> Given a closed curve $\Gamma \subset \mathbb{R}^3$ and a real number V, find the surface of least area with boundary Γ and enclosing a volume V.

The surfaces that are solutions to this problem have constant mean curvature but not necessarily zero mean curvature. Classically there are two ways to approach these problems.

Firstly, we start with the *Plateau problem*. It was J.A.F. Plateau himself who conducted many experiments with soap films and soap bubbles producing minimal and cmc surfaces for a variety of contours [Pla73]. The natural question that arose was, if given a closed curve Γ, there exists a cmc surface spanning Γ and how many such surfaces are there? We point out that the surfaces obtained in the experiments with soap bubbles, i.e., surfaces that are physically realizable, are stable and they are minimizers, at least locally, of the surface area. Mathematically, we look for a parametric surface $X = X(u, v)$ where (u, v) belongs to a given domain $\Omega \subset \mathbb{R}^2$. This fixes the topology of the immersed surface that we are looking for. Usually Ω is the unit disk and for constant mean curvature H, the Plateau problem formulates as follows.

Given a Jordan curve $\Gamma \subset \mathbb{R}^3$ and $H \in \mathbb{R}$, find a parametric surface M spanning Γ such that the mean curvature of M is H. For this, we consider a surface M parametrized on the closure of the unit disk $\Omega = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$, $M = X(\overline{\Omega})$, where $X : \overline{\Omega} \to \mathbb{R}^3$ satisfies

$$\begin{cases} \Delta X = 2H(X_u \times X_v) & \text{on } \Omega, \\ |X_u|^2 - |X_v|^2 = \langle X_u, X_v \rangle = 0 & \text{on } \Omega, \\ X : \partial\Omega \to \Gamma & \text{is a homeomorphism.} \end{cases} \tag{1.2}$$

The works of J. Douglas and T. Radó on minimal surfaces solved the case $H = 0$ [Dou30, Dou31, Dou39a, Dou39b, Rad30a, Rad30b, Rad32]. For his contributions on the Plateau problem, J. Douglas was one of the first two Fields Medallists in 1936. More information on the Plateau problem for minimal surfaces can be found in [Alm66a, Cou50, DHKW92, Nit89, Rad51]. Later, E. Heinz initiated in the fifties the study of the Plateau problem when $H \neq 0$ [Hei54]. There exists a number of existence theorems of (1.2) due to S. Hildebrandt, K. Steffen, H.C. Wente and H. Werner, among others: [Hil70, Ste76, SW78, Wen69, Wen71, Wen80b, Wer57]. See also [GuSp71, GuSp72, HK72]. Surveys on this subject are [BCG02, DuSt96, Ste99, Str88, Str00].

We do not know about the geometry of a solution of the Plateau problem because the surface may have self-intersections, multiple coverings and branch points. Without entering into details, for $H = 0$, it is more useful to replace the area of X, namely, $A(X) = \int_\Omega |X_u \times X_v| \, du \, dv$, by the Dirichlet integral

$$D(X) = \frac{1}{2} \int_\Omega |\nabla X|^2 \, du \, dv, \tag{1.3}$$

since $A(X) \leq D(X)$ and a minimizer of D minimizes the area functional A. If $H \neq 0$, the surface is obtained by minimizing the functional

$$D_H(X) = D(X) + \frac{2H}{3} \int_\Omega \langle X_u \times X_v, X \rangle \, du \, dv \tag{1.4}$$

in a convenient class $\mathscr{C}(\Gamma)$ of immersions defined on the unit disk Ω with boundary Γ. The second term of D_H is, up to the constant $2H$, the algebraic volume with respect to the origin of the surface $X(u, v)$. Notice that the surface is obtained as a solution of a variational problem associated to the Dirichlet integral with an extra term, a Lagrange multiplier, involving the volume of the surface. However, we do not prescribe the volume of the immersions of the class $\mathscr{C}(\Gamma)$ (see also Sect. 2.1).

The results summarize as follows. Given a simple closed curve $\Gamma \subset \mathbb{R}^3$ and H a number sufficiently 'small' in terms of the geometry of Γ, there exists a disk immersed in $\mathbb{R}^3$ with mean curvature H and boundary Γ. The techniques use direct methods of the calculus of variations and we find the minimum of the functional as

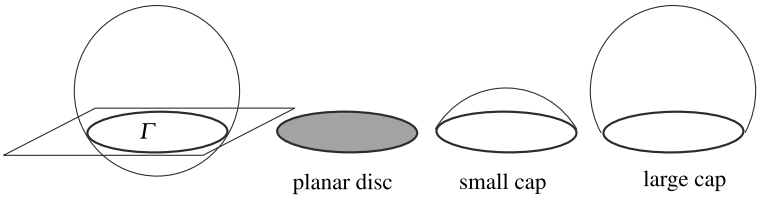

Fig. 1.1 Surfaces with constant mean curvature spanning a given circle Γ obtained by intersecting a sphere with a plane: a planar disk and two spherical caps

the limit of a minimizing sequence. With an appropriate concept of compactness and properties of the functional D_H, such as, lower semicontinuity, we guarantee the existence of a limit, which minimizes D_H. In order to obtain convergent subsequences of minimizing sequences, we consider immersions whose images are included in bounded domains and finally apply the machinery of functional analysis.

For the Plateau problem, all these results have been established by H. Brézis, J.M. Coron, K. Steffen and M. Struwe, who proved the existence of a second solution with the same mean curvature and boundary [BC82, BC84, Ste86, Str85, Str86]. This gives an affirmative answer to a conjecture attributed to Rellich that stated that for $H \neq 0$, there will be at least two geometrically distinct ('small' and 'large') surfaces of constant mean curvature H spanning a given curve: see [HC99, p. 227]. This second solution is usually called a 'large' solution.

H.C. Wente follows an alternative approach in the Plateau problem, prescribing the value V of the volume instead of the mean curvature. The method consists of minimizing the Dirichlet integral D in the class of immersions from the disk enclosing the volume V and with the same boundary. He proves that for each real number V and each simple closed curve Γ, there exists an immersion from the disk into $\mathbb{R}^3$ with constant mean curvature bounded by Γ and with volume V [Wen71].

Even when the boundary Γ of the surface is a circle, we know very little about the family of compact surfaces M immersed in $\mathbb{R}^3$ with constant mean curvature H and with $\partial M = \Gamma$. E. Heinz proved that a necessary condition is $|H| \leq 1/r$, where r is the radius of Γ [Hei69]. If $H = 0$, the maximum principle proves that the flat disk spanning Γ is the only possibility. If $H \neq 0$, two known examples are the spherical caps of constant mean curvature H and with boundary Γ obtained by intersecting a sphere of radius $1/|H|$ with an appropriate plane. From now on, we distinguish both caps, and we call a *small spherical cap* if it is contained in a hemisphere and a *large spherical cap* if it contains a hemisphere as depicted in Fig. 1.1. When both caps agree, we have two equal hemispheres and this occurs when the plane goes through the center of the sphere. Notice that:

1. Flat disks and spherical caps are the only compact umbilical surfaces spanning a circle.
2. Flat disks and spherical caps are the only compact cmc surfaces of revolution spanning a circle.

The last statement is a direct consequence of Delaunay's classification of the family of cmc surfaces of revolution in $\mathbb{R}^3$ [Del41].

Using the construction method of closed cmc surfaces based on balanced diagrams, N. Kapouleas proved the existence of a *non-rotational* compact cmc surface immersed in $\mathbb{R}^3$ spanning a circle [Kap91]. This example has self-intersections, genus greater than 2 and lies in one of the halfspaces determined by the boundary plane. However, at the present time, we do not have any numerical pictures of such examples. This contrasts with the profusion of figures of closed and complete cmc surfaces, such as those found in the previously cited web sites.

Motivated by the above three characterizations of the sphere in the family of closed cmc surfaces, together with the lack of examples, we make the following three conjectures:

Conjecture 1 *A cmc disk immersed in $\mathbb{R}^3$ with circular boundary is umbilical.*

Conjecture 2 *A compact embedded cmc surface in $\mathbb{R}^3$ with circular boundary is umbilical*

Conjecture 3 *A compact stable cmc surface in $\mathbb{R}^3$ with circular boundary is umbilical.*

It is surprising that the boundary versions of the Hopf, Alexandrov and Barbosa-do Carmo theorems for the simplest boundary curve, i.e., a circle, remain as open problems. In this sense, we could say that at present, unlike the sphere, spherical caps cannot be granted their own section of characterizations in the Hilbert and Cohn-Vossen's famous book "Geometry and the Imagination" [HC99, §32].

Another approach to the study of cmc surfaces with boundary is the Dirichlet problem, which roughly says:

Given a closed curve $\Gamma \subset \mathbb{R}^3$ and $H \in \mathbb{R}$, find a graph with constant mean curvature H spanning Γ.

This is the non-parametric approach for finding cmc surfaces with boundary. Of course, Γ must be a graph on a plane curve and so, we view Γ as a function φ defined on the boundary of a planar domain. Explicitly, the problem is as follows:

Given a domain $\Omega \subset \mathbb{R}^2$, $H \in \mathbb{R}$ and $\varphi : \partial\Omega \to \mathbb{R}$, find a smooth function u such that

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 2H \quad \text{on } \Omega \tag{1.5}$$

$$u = \varphi \quad \text{along } \partial\Omega \tag{1.6}$$

where ∇ and div are the Euclidean gradient and divergence operators in $\mathbb{R}^2$.

The graph $z = u(x, y)$ is a surface with constant mean curvature H and its boundary is the graph of φ. Equation (1.5) represents the constant mean curvature equation and it is the expression of the mean curvature H of the surface $z = u(x, y)$.

Independently of the value of φ, there are a priori conditions on the values of H for the solvability of the problem. For example, if Ω is a bounded domain, a simple application of the divergence theorem in (1.5) shows that $|H| \leq L(\partial\Omega)/(2a(\Omega))$, where $L(\partial\Omega)$ and $a(\Omega)$ are the length of $\partial\Omega$ and the area of Ω, respectively. In the case that Ω is a round disk of radius r, then $|H| \leq 1/r$, as occurs for spherical caps. On the other hand, if Ω contains a disk of radius $r > 0$, then necessarily $|H| \leq 1/r$. This is a consequence of the maximum principle comparing spherical caps with the graph of u. In fact, R. Finn proved that if $H \neq 0$ and Ω contains a round disk of radius $1/|H|$, then the graph describes a hemisphere [Fin65].

The first existence result is due to R. Finn in the particular case of minimal graphs, i.e., $H = 0$ in (1.5). Extending previous results of S.N. Bernstein and T. Radó, he showed in 1965 [Fin65]:

> There exists a minimal graph on $\Omega \subset \mathbb{R}^2$ for each arbitrary boundary condition φ if and only if Ω is convex.

See also [Brn10, Fin54, Rad51]. T. Radó showed the sufficiency proving that the solution of the Plateau problem when the prescribed boundary curve Γ is a graph on the boundary $\partial\Omega$ of a convex domain $\Omega \subset \mathbb{R}^2$ must necessarily be a graph on Ω. This was generalized in 1982 by W.H. Meeks III and S.T. Yau proving that if $W \subset \mathbb{R}^3$ is a bounded region whose boundary ∂W has non-negative mean curvature with respect to the inward normal and $\Gamma \subset \partial W$ is a simple closed curve null homotopic in W, then the solution of the Plateau problem is properly embedded [MY82a, MY82b]. In the case of the Radó result, W is the right cylinder $\Omega \times \mathbb{R}$. In addition, T. Radó modified the initial Plateau problem proving the existence of a minimal surface when the prescribed boundary Γ has single valued radial projection onto a plane convex curve [Rad32, Rad51].

For $H \neq 0$, J. Serrin [Ser69b] showed in 1969:

> If $\Omega \subset \mathbb{R}^2$ is a bounded strictly convex domain, then there exists a solution of (1.5)–(1.6) for each arbitrary boundary condition φ if and only if
>
> $$\kappa \geq 2H > 0,$$
>
> where κ is the curvature of the plane curve $\partial\Omega$.

The machinery for solving the Dirichlet problem involves the Leray-Schauder theory for quasilinear elliptic equations of second order. The existence reduces to finding a priori estimates of $\sup_\Omega |u|$ and $\sup_\Omega |\nabla u|$ for a prospective solution u

of (1.5)–(1.6), that is, if there is a constant C, depending only on the initial data H, Ω and φ, such that if u is a solution of (1.5)–(1.6), then

$$|u| + |\nabla u| \leq C$$

on Ω. This assertion seems to be surprising because in order to prove the existence of a solution, we *assume* in principle that such a solution u exists. Then the effort consists of bounding $|u|$ and $|\nabla u|$ by a constant C independently of u. If we are able to find this constant C, then, indeed, *u does exist*.

Other kinds of boundary conditions can be considered to find cmc surfaces but they will not be covered in this book. Among them, it should be pointed out, are boundary conditions of Neumann type, where we replace the prescribing boundary function φ in (1.6) by the angle between the graph of the solution u of (1.5) and the right cylinder $\partial\Omega \times \mathbb{R}$. This is expressed as

$$\left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \mathbf{n} \right\rangle = \cos\gamma,$$

where $\mathbf{n}$ is the inner unit normal along $\partial\Omega$ in $\mathbb{R}^2$. We say that the graph is a *capillary surface*and physically represents the height $z = u(x, y)$ of a liquid that raises in the vertical tube $\Omega \times \mathbb{R}$ by the capillary forces in absence of gravity. The constant γ represents the angle of the surface with the walls of the cylinder. We refer to [Fin86, Lan02].

Chapter 2
Surfaces with Constant Mean Curvature

In this chapter we shall review some basic aspects of the theory of surfaces with constant mean curvature. Rather the reader should take this chapter as a first introduction to the problems and as a way to become acquainted with the methods that will be needed in successive chapters. In the process of this review, we shall obtain results on compact cmc surfaces with boundary. The surfaces with constant mean curvature will arise as solutions of a variational problem associated to the area functional and we shall relate it to the classical isoperimetric problem. Then we state the first and second variation formula for the area and we give the notion of stability of a cmc surface. Next, we shall introduce complex analysis as a basic tool in the theory and we define the Hopf differential. This will allow us to prove the Hopf theorem. In addition, we compute the Laplacians of some functions that contain geometric information of a cmc surface. As the Laplacian is an elliptic operator, we are in position to apply the maximum principle and we will obtain height estimates of a graph of constant mean curvature. Finally, and with the aid of the expression of these Laplacians, we will derive the Barbosa-do Carmo theorem that characterizes a round sphere within the family of closed stable cmc surfaces of Euclidean space.

2.1 The Variational Formula for the Area

Let M be a connected orientable (smooth) surface with possibly non-empty boundary ∂M. Denote by $int(M) = M \setminus \partial M$ the set of interior points of M. Let $x : M \to \mathbb{R}^3$ be an immersion of M in Euclidean three-space $\mathbb{R}^3$. We say that M is immersed in $\mathbb{R}^3$ if the immersion x is assumed. If $p \in M$, we write p instead of $x(p)$, or simply, x. We represent by $N : M \to \mathbb{S}^2$ a Gauss map (or an orientation), where $\mathbb{S}^2$ denotes the unit sphere of $\mathbb{R}^3$. An immersion x is called an embedding if $x : M \to x(M)$ is a homeomorphism and we say that the surface M is embedded in $\mathbb{R}^3$. If M is compact, this is equivalent to $x(M)$ having no self-intersections and we will identify the surface M with its image $x(M)$.

Let $\Gamma \subset \mathbb{R}^3$ be a (smooth) space curve. Consider an immersion $x : M \to \mathbb{R}^3$ of a surface M. We say that Γ is the boundary of the immersion x if $x_{|\partial M} : \partial M \to \Gamma$ is

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_2,
© Springer-Verlag Berlin Heidelberg 2013

a diffeomorphism. In particular, we prohibit that $x_{|\partial M}$ turns many times around Γ. We also say that x (or M) spans Γ or that Γ is the boundary of $x(M)$. If x is an embedding we will use interchangeably ∂M and Γ.

We endow M with the Euclidean metric $\langle \cdot, \cdot \rangle$ induced by the immersion x. If M is a compact surface, the areaof M is defined by

$$A = \int_M dM,$$

where dM denotes the area element of M. Now we introduce the volume of the immersion and we motivate the definition by considering M a closed embedded surface. In such case, M defines a bounded 3-domain $W \subset \mathbb{R}^3$. We call the volume V enclosed by M the Lebesgue volume of W. An expression of V in terms of an integral on M may be derived by an easy application of the divergence theorem. The divergence of the vector field $Y(x, y, z) = (x, y, z)$ is 3 and thus

$$V = \int_{\overline{W}} 1 = \frac{1}{3} \int_{\overline{W}} \mathrm{Div}(Y) = -\frac{1}{3} \int_M \langle N, Y \rangle \, dM = -\frac{1}{3} \int_M \langle N, x \rangle \, dM,$$

where N is the Gauss map on M that points towards W. In general, if $x : M \to \mathbb{R}^3$ is an immersion of a compact surface M with possibly non-empty boundary, the *algebraic volume* of x with respect to the orientation N is defined by

$$V = -\frac{1}{3} \int_M \langle N, x \rangle \, dM. \tag{2.1}$$

When $\partial M \neq \emptyset$, the number V measures the algebraic volume determined by the surface together with the cone C formed by ∂M and the origin of $\mathbb{R}^3$. This follows if we parametrize C as $\phi(s, t) = t\alpha(s)$, with $\alpha : \partial M \to x(\partial M)$ a parametrization of $x(\partial M)$ and $0 \leq t \leq 1$. Since C together with M forms a 2-cycle, the same vector field Y gives

$$3V = -\int_M \langle N, x \rangle \, dM - \int_C \langle N_C, \phi \rangle \, dC,$$

where N_C is the unit normal vector to C. Finally, observe that $\langle N_C, \phi \rangle = 0$.

We remark that the volume of the immersion depends on the origin, or in other words, the number V changes by translations of the initial surface. However, when the boundary is planar, and as a consequence of the divergence theorem again, we have:

Proposition 2.1.1 *Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M. Assume that $x(\partial M) = \Gamma$ is contained in a plane P. Then the volume is invariant provided the origin lies in P.*

A particular case occurs when M is the graph of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain and $u = 0$ on $\partial\Omega$. Choose on M the usual

parametrization of a graph, namely, $\Psi(x, y) = (x, y, u(x, y))$ and the orientation given by

$$N\big(x, y, u(x, y)\big) = \frac{1}{\sqrt{1 + u_x^2 + u_y^2}}(-u_x, -u_y, 1). \tag{2.2}$$

As $dM = \sqrt{1 + u_x^2 + u_y^2}\, dxdy$, we have

$$V = -\frac{1}{3}\int_M \langle N, \Psi\rangle\, dM = \frac{1}{3}\int_\Omega (xu_x + yu_y - u)\, dx\, dy.$$

Define on Ω the vector field $Y(q) = u(q)q$. Then $\mathrm{Div}(Y) = xu_x + yu_y + 2u$ and using that $u = 0$ on $\partial\Omega$, we find that

$$\int_\Omega (xu_x + yu_y + 2u)\, dx\, dy = -\int_{\partial\Omega} u\langle \mathbf{n}, q\rangle\, ds = 0,$$

where $\mathbf{n}$ is the inward conormal vector of Ω along $\partial\Omega$. We conclude that

$$V = -\int_\Omega u\, dx\, dy. \tag{2.3}$$

As was noted in Chap. 1, the mean curvature of the surface can be motivated if we consider the so-called isoperimetric problem.

Isoperimetric problem: Among all compact surfaces in Euclidean space $\mathbb{R}^3$ enclosing a given volume, find the surface of least area.

If we modify the question and we only ask for those surfaces that are solutions of the isoperimetric problem up to the first order, the problem becomes the following:

Variational problem: Characterize a compact surface in Euclidean space $\mathbb{R}^3$ whose area is critical among all variations that preserve the volume of the surface.

Definition 2.1.2 A variation of an immersion $x : M \to \mathbb{R}^3$ is a differentiable map $X : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ such that for each $t \in (-\varepsilon, \varepsilon)$, the maps $x_t : M \to \mathbb{R}^3$ given by $x_t(p) = X(p, t)$ are immersions for all t, and for $t = 0$, $x_0 = x$. The variational vector field of the variation $\{x_t\}$ is defined by

$$\xi(p) = \left.\frac{\partial X(p, t)}{\partial t}\right|_{t=0}, \quad p \in M.$$

The variation $\{x_t\}$ is said to be *admissible* if preserves the boundary of x, that is, $x_t(p) = x(p)$ for every $p \in \partial M$. In particular, $\xi = 0$ on ∂M. Assume M is a compact surface. Consider the area $A(t)$ and the volume $V(t)$ of M induced by the immersion x_t:

$$A(t) = \int_M dM_t, \qquad V(t) = -\frac{1}{3}\int_M \langle N_t, x_t\rangle\, dM_t,$$

where dM_t and N_t stand for the area element of M induced by x_t and the unit normal field of x_t, respectively. In particular, $A(0)$ and $V(0)$ are the area and the volume of the initial immersion x. We obtain the variation formula for $A(t)$ and $V(t)$ at $t = 0$.

Proposition 2.1.3 (First variation for the area) *The area functional $A(t)$ is differentiable at $t = 0$ and*

$$A'(0) = -2\int_M H\langle N, \xi\rangle\, dM - \int_{\partial M} \langle \nu, \xi\rangle\, ds, \tag{2.4}$$

where ν is the inward unit conormal vector of M along ∂M and H is the mean curvature of the immersion.

The *mean curvature* H is defined by

$$H(p) = \frac{\kappa_1(p) + \kappa_2(p)}{2}, \quad p \in M,$$

where $\kappa_1(p)$ and $\kappa_2(p)$ are the principal curvatures of x at p, i.e., the eigenvalues of the Weingarten endomorphism $A_p = -(dN)_p : T_pM \to T_pM$. In particular, the sign of H changes by reversing the orientation N. On the other hand, the second fundamental form of the immersion is

$$\sigma_p(u, v) = -\big\langle (dN)_p(u), v\big\rangle, \quad u, v \in T_pM.$$

Then the value $2H(p)$ agrees with the trace of σ_p. Recall that the norm $|\sigma|$ of σ is $|\sigma|^2 = 4H^2 - 2K$, where $K = \kappa_1\kappa_2$ is the Gauss curvature of the surface. Moreover, $|\sigma|^2 \geq 2H^2$ on M and equality holds at a point p if and only if p is umbilical.

Proof of Proposition 2.1.3 A proof of (2.4) appears in Proposition A.0.1, Appendix A. Here we restrict to the case when the variational vector field is perpendicular to the surface. Let $x_t : M \to \mathbb{R}^3$ be the variation of x given by $x_t(p) = x(p) + tf(p)N(p)$, $t \in (-\varepsilon, \varepsilon)$ with $f \in C^\infty(M)$. The variational vector field is normal to the surface because $\xi = fN$. A computation leads to

$$(dx_t)_p(v) = v + tf(p)\, dN_p(v) + t df_p(v)N(p),$$

for $v \in T_pM$. By the compactness of M, we can choose $\varepsilon > 0$ sufficiently small in order to obtain $(dx_t)_p(v) \neq 0$ for all tangent vectors v and thus, x_t is an immersion. Let e_i denote the principal directions at $p \in M$. Then

$$(dx_t)_p(e_i) = \big(1 - t\kappa_i(p)f(p)\big)e_i + t df_p(e_i)N(p).$$

The Jacobian of x_t is

$$\begin{aligned}\mathrm{Jac}(x_t)(p) &= \left|(dx_t)_p(e_1) \times (dx_t)_p(e_2)\right| \\ &= \left|-t\big(1 - t\kappa_2(p)f(p)\big)df_p(e_1)e_1 - t\big(1 - t\kappa_1(p)f(p)\big)df_p(e_2)e_2\right. \\ &\quad \left. \times \big(1 - 2tH(p)f(p) + t^2K(p)f(p)^2\big)N(p)\right|.\end{aligned}$$

Hence it follows that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Jac}(x_t)(p) = -2H(p)f(p). \tag{2.5}$$

Finally, by (2.5) and the formula for change of variables, we find that

$$A'(0) = \int_M \left.\frac{d}{dt}\right|_{t=0} \mathrm{Jac}(x_t)\, dM = -2\int_M Hf\, dM = -2\int_M H\langle N, \xi\rangle\, dM,$$

showing (2.4). Observe that the boundary integral in (2.4) vanishes because $\xi(p) \perp T_pM$. □

We say that M is a *surface with constant mean curvature* if the function H is constant. We abbreviate this by saying that M is a *cmc surface* or an *H-surface* if we want to emphasize the value H of the mean curvature. In the particular case that $H = 0$ on M, we say that M is a *minimal surface*.

In view of Proposition 2.1.3, we obtain the following characterization of a minimal surface:

Theorem 2.1.4 *Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M. Then the immersion is minimal if and only if $A'(0) = 0$ for any admissible variation of x.*

Proof If $H = 0$, then (2.4) gives immediately $A'(0) = 0$. Conversely, let $f \in C^\infty(M)$ be a function with $f > 0$ in $int(M)$ and $f = 0$ on ∂M. Define the variation $x_t(p) = x(p) + tf(p)H(p)N(p)$, which is admissible and $\xi = fHN$. Then (2.4) yields

$$0 = A'(0) = -2\int_M H\langle N, \xi\rangle\, dM = -2\int_M fH^2\, dM.$$

As f is positive on $int(M)$, then $H = 0$ on M. □

This theorem establishes that *a compact surface is minimal if and only it is a critical point of the area for any admissible variation*. As a consequence, if Γ is a closed curve and M is a surface of least area among all surfaces spanning Γ, then M is a minimal surface since for any variation of M, the functional $A(t)$ has a minimum at $t = 0$, hence, $A'(0) = 0$. The reverse process does not hold in general and there exist compact minimal surfaces that are not minimizers. A special case is studied in Proposition 2.1.8 below.

Proposition 2.1.5 (First variation for the volume) *The volume functional $V(t)$ is differentiable at $t = 0$ and*

$$V'(0) = -\int_M \langle N, \xi \rangle \, dM. \tag{2.6}$$

For a proof, see Proposition A.0.2, Appendix A. Once the formulas for the first variation for the area and for the volume have been obtained, we have the conditions for characterizing a surface with constant mean curvature. If we recall the motivation of cmc surfaces by soap bubbles given in the preceding chapter, we require that the perturbations made on the soap bubble keep the enclosed volume of air constant. Thus we give the next definition. A variation is said to be *volume preserving* if the functional $V(t)$ is constant. In particular, $V'(t) = 0$. Comparing with the proof of Theorem 2.1.4 for minimal surfaces and viewing the integral in (2.6), for a cmc surface it does not suffice that $f = 0$ on ∂M. For this reason, we need the following result [BC84].

Lemma 2.1.6 *Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M and let $f \in C^\infty(M)$ with $\int_M f \, dM = 0$. Then there exists a volume preserving variation whose variational vector field is $\xi = fN$. Furthermore, if $f = 0$ on ∂M, then the variation can be assumed admissible.*

Proof Let g be a differentiable function on M such that $g = 0$ on ∂M and $\int_M g \, dM \neq 0$. If $I = (-\varepsilon, \varepsilon)$, define

$$X : M \times I \times I \to \mathbb{R}^3, \quad X(p, t, s) = x(p) + \big(tf(p) + sg(p)\big)N(p).$$

For $\varepsilon > 0$ sufficiently small, the function X may be seen as a variation of x fixing s or fixing t, with $X(p, 0, 0) = x(p)$. Let $V(t, s)$ be the volume of the surface $X(-, t, s) : M \to \mathbb{R}^3$ and consider the equation $V(t, s) = c$, where c is a constant. By (2.6),

$$\left.\frac{\partial V(t, s)}{\partial s}\right|_{(t,s)=(0,0)} = -\int_M g \, dM \neq 0.$$

The implicit function theorem guarantees the existence of a diffeomorphism $\varphi : I_1 \to I_2$, where I_1 and I_2 are open intervals around 0, such that $\varphi(0) = 0$ and $V(t, \varphi(t)) = c$ for all $t \in I_1$. This allows us to consider the volume preserving variation of x given by $x_t(p) = X(p, t, \varphi(t))$. We show that $\xi = fN$. The derivative of $V(t, \varphi(t)) = c$ with respect to t yields

$$0 = \frac{\partial V}{\partial t}(0) + \varphi'(0)\frac{\partial V}{\partial s}(0) = -\int_M \big(f + \varphi'(0)g\big) \, dM = -\varphi'(0)\int_M g \, dM.$$

This says that $\varphi'(0) = 0$. Thus

$$\xi(p) = \frac{\partial x_t}{\partial t}\bigg|_{t=0} = \big(f(p) + \varphi'(0)g(p)\big)N(p) = f(p)N(p).$$

In the particular case that $f = 0$ along ∂M, and since $g = 0$ on ∂M, we have $x_t(p) = x(p)$ along ∂M and thus the variation is admissible. □

Theorem 2.1.7 *Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M. Then x has constant mean curvature if and only if $A'(0) = 0$ for all volume preserving admissible variations.*

Proof Assume that the mean curvature H is constant. For a volume preserving variation of x, $V'(t) = 0$. In addition, if the variation is admissible, $\xi = 0$ on ∂M. Since H is constant, the expression of $A'(0)$ in (2.4) gives

$$A'(0) = -2H \int_M \langle N, \xi \rangle \, dM = 2HV'(0) = 0.$$

Conversely, assume that x is a critical point of the area for any volume preserving admissible variation. Let

$$H_0 = \frac{1}{A_0} \int_M H \, dM,$$

where A_0 is the area of M and define the function $f = H - H_0$. Observe that H is constant if and only if $f = 0$. By contradiction, assume that at an interior point, $f \neq 0$. As $\int_M f dM = 0$, the sets $M^+ = \{p \in int(M) : f(p) > 0\}$ and $M^- = \{p \in int(M) : f(p) < 0\}$ are non-empty. Fix points $p^+ \in M^+$, $p^- \in M^-$ and consider the corresponding bump functions $\varphi^+, \varphi^- : M \to \mathbb{R}$, that is, smooth non-negative functions with $\text{supp}(\varphi^+) \subset M^+$, $\text{supp}(\varphi^-) \subset M^-$ and $\varphi^+(p^+) = \varphi^-(p^-) = 1$. In particular, $\varphi^+, \varphi^- = 0$ on ∂M. Let

$$\alpha^+ = \int_M \varphi^+ f \, dM > 0, \qquad \alpha^- = \int_M \varphi^- f \, dM < 0$$

and $\lambda > 0$ be the real number such that $\alpha^+ + \lambda\alpha^- = 0$. Define $\varphi = \varphi^+ + \lambda\varphi^-$. This non-zero function satisfies $\varphi \geq 0$ on $int(M)$, $\varphi = 0$ on ∂M and

$$\int_M \varphi f \, dM = \int_{M^+} \varphi^+ f \, dM + \lambda \int_{M^-} \varphi^- f \, dM = 0. \tag{2.7}$$

From the preceding lemma, there exists an admissible variation of x that preserves the volume and whose variational vector field is $\xi = \varphi f N$. Then (2.4) and (2.7) now give

$$0 = A'(0) = -2 \int_M H\varphi f \, dM = -2 \int_M (H - H_0)\varphi f \, dM = -2 \int_M \varphi f^2 \, dM.$$

Thus $\varphi f^2 = 0$ on M. However, at the point p^+, $(\varphi f)^2(p^+) = f(p^+)^2 > 0$, a contradiction. □

For cmc graphs, we have:

Proposition 2.1.8 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let M be a compact cmc graph on Ω. Then M minimizes the area among all graphs spanning ∂M and with the same volume as M.*

Proof Suppose that M is an H-graph of a function u defined on Ω and let M' be another graph with $\partial M' = \partial M$ and the same volume as M. Then $M \cup M'$ determines an oriented 3-domain $W \subset \mathbb{R}^3$ with zero volume V because the volumes of M and M' agree. On W define the vector field $Y(x, y, z) = N(x, y, u(x, y))$, where N is the unit normal field of M according to (2.2). We compute the divergence of Y. Taking into account (1.5) we see that

$$\begin{aligned}\operatorname{Div}(Y) &= \partial_x\left(\frac{-u_x}{\sqrt{1+|\nabla u|^2}}\right) + \partial_y\left(\frac{-u_y}{\sqrt{1+|\nabla u|^2}}\right) + \partial_z\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right)\\ &= -\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = -2H.\end{aligned}$$

The divergence theorem gives

$$\begin{aligned}0 = -2HV &= \int_W \operatorname{Div}(Y) = \int_M \langle Y, N\rangle\, dM + \int_{M'} \langle Y, N'\rangle\, dM'\\ &= \operatorname{area}(M) + \int_{M'} \langle Y, N'\rangle\, dM',\end{aligned}$$

where N' is the unit normal of M' pointing outwards from W. Hence, and as $|\langle Y, N'\rangle| \le 1$, we conclude

$$\operatorname{area}(M) = -\int_{M'} \langle Y, N'\rangle\, dM' \le \int_{M'} dM' = \operatorname{area}(M').$$ □

In fact, a more general result asserts that a minimal graph over a convex domain is area-minimizing among all compact surfaces with the same boundary [Fed69, Mor95]. Since a surface is locally the graph of a function defined in a convex domain, then we conclude that *a minimal surface locally minimizes area*.

There exists another variational characterization of a surface with constant mean curvature using Lagrange multipliers. Given a (not necessarily volume preserving) admissible variation X and $\lambda \in \mathbb{R}$, define the functional J_λ as

$$J_\lambda(t) = A(t) - 2\lambda V(t), \quad t \in (-\varepsilon, \varepsilon). \tag{2.8}$$

By (2.4) and (2.6),

$$J'_\lambda(0) = -2\int_M (H-\lambda)\langle N, \xi\rangle \, dM. \tag{2.9}$$

Theorem 2.1.9 *Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M. Then x has constant mean curvature H if and only if $J'_H(0) = 0$ for all admissible variations.*

Proof If x has constant mean curvature H, take $\lambda = H$ in (2.9). To prove the converse, assume that there exists a $\lambda \in \mathbb{R}$ such that $J'_\lambda(0) = 0$ for any admissible variation. In particular, it holds for any volume preserving admissible variation $\{x_t\}$, and since λ is a constant, we have

$$0 = J'_\lambda(0) = A'(0) - 2\lambda V'(0) = A'(0).$$

Since $A'(0) = 0$ holds for all volume preserving admissible variation, we apply Theorem 2.1.7 concluding that H is constant. □

This characterization of a cmc surface is related to the Plateau problem (1.2), where the solutions are obtained by minimizing D_H in a suitable space of immersions from a disk into $\mathbb{R}^3$, as was remarked in Chap. 1. Comparing the expressions of J_λ and D_H in (2.8) and (1.4) respectively, the area integral is replaced by the Dirichlet integral and the volume V by its formulation for a parametric surface.

Among all critical points of the area, we consider those that are local minimizers. Then it is natural to study the second variation of the area because for a surface of least area, the second derivative $A''(0)$ is not negative. Since a surface with constant mean curvature is characterized in variational terms by Theorems 2.1.7 and 2.1.9, there are two different notions of stability.

Definition 2.1.10 Let M be a compact surface and let $x : M \to \mathbb{R}^3$ be an immersion of constant mean curvature H.

1. The immersion x is said to be stable if $A''(0) \geq 0$ for all volume preserving admissible variations.
2. The immersion x is said strongly stable if $J''_H(0) \geq 0$ for all admissible variations.

With respect to the first definition, some authors prefer to use the term 'volume preserving stable' rather than 'stable'.

Proposition 2.1.11 (Second variation for the area) *Let M be a compact surface and let $x : M \to \mathbb{R}^3$ be an immersion of constant mean curvature H. If $\{x_t\}$ is a volume preserving admissible variation with $f = \langle N, \xi\rangle$, then*

$$A''(0) = -\int_M f\left(\Delta f + |\sigma|^2 f\right) dM. \tag{2.10}$$

Here Δ denotes the Laplacian-Beltrami operator on M.

For a proof, see Corollary A.0.4 in Appendix A. The term in the parentheses of (2.10), namely, $L(f) = \Delta f + |\sigma|^2 f$, is called the *Jacobi operator* of the immersion. As a consequence, the solutions of the isoperimetric problem are stable because they are minimizers of the area.

Similarly, the formula for the second variation of the function J_H is

$$J_H''(0) = -\int_M f\big(\Delta f + |\sigma|^2 f\big)\, dM,$$

for all admissible variation of the immersion. Here the function f only satisfies $f = 0$ along ∂M.

Let f be a smooth function on M with $f = 0$ on ∂M. Since $\operatorname{div}(f\nabla f) = f\Delta f + |\nabla f|^2$, the divergence theorem changes the expression of $A''(0)$ into

$$A''(0) = \int_M \big(|\nabla f|^2 - |\sigma|^2 f^2\big)\, dM. \tag{2.11}$$

This allows us to extend the right-hand side of (2.11) as a quadratic form I in the Sobolev space $H_0^{1,2}(M)$, the completion of $C_0^\infty(M)$ in $L^2(M)$:

$$I : H_0^{1,2}(M) \to \mathbb{R}, \qquad I(f) = \int_M \big(|\nabla f|^2 - |\sigma|^2 f^2\big)\, dM,$$

where now ∇f denotes the weak gradient of f. Therefore, a surface M is:

1. stable if $I(f) \geq 0$ for all $f \in H_0^{1,2}(M)$ such that $\int_M f\, dM = 0$;
2. strongly stable if $I(f) \geq 0$ for all $f \in H_0^{1,2}(M)$.

A first example of a stable surface is the sphere.

Proposition 2.1.12 *A round sphere is stable.*

Proof Assume without loss of generality that the sphere is $\mathbb{S}^2$. Consider the spectrum of the Laplacian operator Δ. The first eigenvalue is $\lambda_1 = 0$ and the eigenfunctions are the constant functions. The second eigenvalue is $\lambda_2 = 2$ [CH89]. By the Rayleigh characterization of λ_2,

$$2 = \lambda_2 = \min\left\{ \frac{\int_{\mathbb{S}^2} |\nabla f|^2\, d\mathbb{S}^2}{\int_{\mathbb{S}^2} f^2\, d\mathbb{S}^2} : f \in C^\infty\big(\mathbb{S}^2\big), \int_{\mathbb{S}^2} f\, d\mathbb{S}^2 = 0 \right\}.$$

Hence $\int_{\mathbb{S}^2} |\nabla f|^2\ d\mathbb{S}^2 \geq 2\int_{\mathbb{S}^2} f^2\ d\mathbb{S}^2$ for all differentiable function f with $\int_M f\, dM = 0$. Since $|\sigma|^2 = 2$ on $\mathbb{S}^2$, this proves the stability of $\mathbb{S}^2$. □

Proposition 2.1.13 *A planar disk and a spherical cap are stable. Moreover, a small spherical cap and a hemisphere are strongly stable but a large spherical cap is not strongly stable.*

Proof If M is a planar disk, then $|\sigma|^2 = 0$ and $I(f) = \int_M |\nabla f|^2 \, dM \geq 0$, showing that M is strongly stable. Consider now a spherical cap M. Without loss of generality, assume that the radius of the spherical cap is 1. Denote by S^+ a hemisphere of $\mathbb{S}^2$.

We use the spectrum of the Laplacian Δ with Dirichlet conditions, that is, the solutions of the eigenvalue problem $\Delta f + \lambda f = 0$ on M, $f = 0$ on ∂M. The first eigenvalue for the Dirichlet problem in S^+ is $\lambda_1(S^+) = 2$. Now we employ the monotonicity property of λ_1: if $M_1 \subsetneq M_2$, then $\lambda_1(M_1) > \lambda_1(M_2)$ [CH89].

1. Consider $f \in C^\infty(M)$ with $f = 0$ on ∂M and $\int_M f \, dM = 0$. Extending f to $\mathbb{S}^2$ by 0, we obtain a function $\tilde{f} \in H^{1,2}(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} \tilde{f} \, d\mathbb{S}^2 = 0$. As $\mathbb{S}^2$ is stable by the preceding proposition, $I(f) = I(\tilde{f}) \geq 0$.
2. Since a small spherical cap M is included in a hemisphere S^+, $\lambda_1(M) > \lambda_1(S^+) = 2$. Thus if M is a small spherical cap or a hemisphere, the variational characterization of $\lambda_1(M)$ gives

$$2 \leq \lambda_1(M) \leq \frac{\int_M |\nabla f|^2 \, dM}{\int_M f^2 \, dM}$$

for all $f \in C_0^\infty(M)$. As $|\sigma|^2 = 2$ on M, we conclude $I(f) \geq 0$.

For a large spherical cap M, $\lambda_1(M) < \lambda_1(S^+) = 2$. If f is an eigenfunction of $\lambda_1(M)$, then

$$\frac{\int_M |\nabla f|^2 \, dM}{\int_M f^2 \, dM} = \lambda_1(M) < 2,$$

so $I(f) < 0$ and M is not strongly stable. □

As we observe in the above proof, strongly stability is related to the eigenvalue problem associated with the Jacobi operator $L = \Delta + |\sigma|^2$, that is,

$$\begin{cases} L(f) + \lambda f = 0 & \text{on } M \\ f = 0 & \text{on } \partial M. \end{cases} \tag{2.12}$$

The operator L is elliptic and thus its spectrum has many properties (see Lemma 8.1.3). For example, the first eigenvalue $\lambda_1(L)$ is characterized by

$$\lambda_1(L) = \min \left\{ \frac{-\int_M f L(f) \, dM}{\int_M f^2 \, dM} : f \in C^\infty(M), \, f = 0 \text{ on } \partial M \right\}.$$

Then we have immediately from (2.10):

Corollary 2.1.14 *A compact cmc surface is strongly stable if and only if* $\lambda_1(L) \geq 0$.

2.2 The Hopf Differential

In this section we associate to each cmc surface a holomorphic 2-form, the so-called Hopf differential, that informs us about the distribution of the umbilical points on the surface.

Let $x : M \to \mathbb{R}^3$ be an immersion of an orientable surface M. We use the classical notation $\{E, F, G\}$ and $\{e, f, g\}$ for the coefficients of the first and second fundamental forms, respectively, in local coordinates $x = x(u, v)$. Consider conformal coordinates (u, v) defined on an open subset V of M, i.e.,

$$E = \langle x_u, x_u \rangle = G = \langle x_v, x_v \rangle, \qquad F = \langle x_u, x_v \rangle = 0.$$

Take the Gauss map

$$N = \frac{x_u \times x_v}{|x_u \times x_v|},$$

where $\times$ is the cross product of $\mathbb{R}^3$. The first and the second fundamental forms are, respectively,

$$\langle \cdot, \cdot \rangle = E\big(du^2 + dv^2\big), \qquad \sigma = edu^2 + 2fdudv + gdv^2,$$

where

$$e = \langle N, x_{uu} \rangle, \qquad f = \langle N, x_{uv} \rangle, \qquad g = \langle N, x_{vv} \rangle.$$

The mean curvature H is given by

$$H = \frac{e + g}{2E}. \tag{2.13}$$

In terms of the Christoffel symbols, the second derivatives of x are:

$$\begin{aligned} x_{uu} &= \frac{E_u}{2E} x_u - \frac{E_v}{2E} x_v + eN \\ x_{uv} &= \frac{E_v}{2E} x_u + \frac{E_u}{2E} x_v + fN \\ x_{uu} &= -\frac{E_u}{2E} x_u + \frac{E_v}{2E} x_v + gN. \end{aligned} \tag{2.14}$$

On the other hand,

$$N_u = -\frac{e}{E} x_u - \frac{f}{E} x_v, \qquad N_v = -\frac{f}{E} x_u - \frac{g}{E} x_v. \tag{2.15}$$

Using (2.13), (2.14) and (2.15), the Codazzi equations are

$$\begin{aligned} e_v - f_u &= \langle N_v, x_{uu} \rangle - \langle N_u, x_{uv} \rangle = \frac{E_v}{2E}(e + g) = E_v H, \\ f_v - g_u &= \langle N_v, x_{uv} \rangle - \langle N_u, x_{vv} \rangle = -\frac{E_u}{2E}(e + g) = -E_u H. \end{aligned}$$

By differentiating $2EH = e + g$ with respect to u and v, we have

$$2E_u H + 2EH_u = e_u + g_u, \qquad 2E_v H + 2EH_v = e_v + g_v.$$

With both expressions, the Codazzi equations now read as

$$(e - g)_u + 2f_v = 2EH_u, \qquad (e - g)_v - 2f_u = -2EH_v, \tag{2.16}$$

respectively. Let us introduce the complex notation $z = u + iv$, $\bar{z} = u - iv$ and

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v), \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v).$$

Define

$$\Phi(z, \bar{z}) = e - g - 2if.$$

Equations (2.16) are then simplified as

$$\Phi_{\bar{z}} = EH_z. \tag{2.17}$$

We point out that the zeroes of Φ are the umbilical points of the immersion since $\Phi(p) = 0$ if and only if $e = g$ and $f = 0$ at p. We express Φ as follows. Let us consider the derivatives x_z and N_z:

$$x_z = \frac{1}{2}(x_u - ix_v), \qquad N_z = \frac{1}{2}(N_u - iN_v).$$

Then

$$\langle x_z, N_z \rangle = -\frac{1}{4}(e - g - 2if) = -\frac{1}{4}\Phi(z, \bar{z}), \tag{2.18}$$

and thus

$$\Phi(z, \bar{z}) = -4\langle x_z, N_z \rangle.$$

We study Φ under a change of conformal coordinates $w = h(z)$, where h is a holomorphic function. Then $x_z = h'(z)x_w$, $N_z = h'(z)N_w$ and

$$\Phi(z, \bar{z}) = -4\langle x_z, N_z \rangle = -4h'(z)^2\langle x_w, N_w \rangle = h'(z)^2\Phi(w, \bar{w}).$$

Hence

$$\Phi(w, \bar{w})dw^2 = \Phi(w, \bar{w})h'(z)^2dz^2 = \Phi(z, \bar{z})dz^2.$$

This equality means that Φdz^2 defines a global quadratic differential form on the surface M.

Definition 2.2.1 The differential form Φdz^2 is called the Hopf differential.

Theorem 2.2.2 *A conformal immersion $x : M \to \mathbb{R}^3$ has constant mean curvature if and only if $\Phi(dz)^2$ is holomorphic. In such case, either the set of umbilical points is formed by isolated points, or the immersion is umbilical.*

Proof Given a conformal parametrization $x(u, v)$, Eq. (2.17) gives $\Phi_{\bar{z}} = 0$, which is equivalent to saying that Φ is holomorphic on V. Thus the umbilical points agree with the zeroes of a holomorphic function. This means either $\Phi = 0$ on V or the umbilical points are isolated. In the first case, an argument of connectedness proves that the set of umbilical points is an open and closed set of M and so M is an umbilical surface. □

Theorem 2.2.2 is also valid in hyperbolic space $\mathbb{H}^3$ and for the sphere $\mathbb{S}^3$. A direct consequence is the Hopf theorem [Hop83]:

Theorem 2.2.3 (Hopf) *The only closed cmc surface of genus* 0 *in the space forms $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{S}^3$ is the standard sphere.*

Proof Since the genus of M is zero, the uniformization theorem says that its conformal structure is conformally equivalent to the usual structure of $\overline{\mathbb{C}}$. This is defined by the parametrizations z in $\mathbb{C}$ and $w = 1/z$ in $\overline{\mathbb{C}} - \{0\}$. The intersection domain of both charts is $\mathbb{C} - \{0\}$ and in this open set, the Hopf differential Φ is

$$\Phi(z) = w'(z)^2 \Phi(w) = \frac{1}{z^4} \Phi(w).$$

It follows that

$$\lim_{z \to \infty} \Phi(z) = \lim_{z \to \infty} \left(\frac{1}{z^4} \right) \Phi(w = 0) = 0.$$

This shows that by writing $\Phi = \Phi(z)$ in terms of the parametrization z on $\mathbb{C}$, Φ can be extended to ∞ by letting $\Phi(\infty) = 0$. Therefore Φ is a bounded holomorphic function on $\overline{\mathbb{C}}$. Liouville's theorem asserts that the only bounded holomorphic functions on $\overline{\mathbb{C}}$ are constant. As $\Phi(\infty) = 0$, then $\Phi = 0$ on $\overline{\mathbb{C}}$ and all points are umbilical. Finally, the only umbilical closed surface in a space form is the sphere. □

2.3 Elliptic Equations for cmc Surfaces

Let $x : M \to \mathbb{R}^3$ be an immersion of a surface M and let N be the Gauss map. Denote by $\mathfrak{X}(M)$ the space of tangent vector fields on M. Recall the Gauss and Weingarten formulae:

$$\nabla^0_X Y = \nabla_X Y + \sigma(X, Y) N, \tag{2.19}$$

$$\nabla^0_X N = -A_N X = -AX, \qquad \sigma(X, Y) = \langle AX, Y \rangle, \tag{2.20}$$

for $X, Y \in \mathfrak{X}(M)$. Here ∇^0 is the usual derivation of $\mathbb{R}^3$, ∇ is the induced connection on M and A is the Weingarten map.

We calculate the Laplacian of the position vector x and the Gauss map N. In order to compute these Laplacians, we use the fact that, fixing $p \in M$, there exists a *geodesic frame around* p, that is, an orthonormal frame $\{e_1, e_2\}$ around p such that at p, $(\nabla_{e_i} e_j)_p = 0$ (for a proof see, for instance, [Car92, Chv94]). Then $(\nabla_{e_i} e_j)_p$ is orthogonal to M at p and if f is a smooth function on M, the Laplacian of f is

$$\Delta f(p) = \sum_{i=1}^{2} e_i \langle e_i, \nabla f \rangle = \sum_{i=1}^{2} e_i \big(e_i(f)\big). \tag{2.21}$$

The next result is valid for any immersion without the need that H is constant.

Proposition 2.3.1 *Let $x : M \to \mathbb{R}^3$ be an immersion of a surface M. If $a \in \mathbb{R}^3$, then*

$$\Delta \langle x, a \rangle = 2H \langle N, a \rangle \tag{2.22}$$

$$\Delta |x|^2 = 4 + 4H \langle N, x \rangle. \tag{2.23}$$

Proof Fix $p \in M$ and let $\{e_1, e_2\}$ be a geodesic frame around p. Using (2.19) and (2.21), we obtain

$$\begin{aligned} \Delta \langle x, a \rangle &= \sum_{i=1}^{2} e_i \big(e_i \langle x, a \rangle\big) = \sum_{i=1}^{2} e_i \langle e_i, a \rangle = \sum_{i=1}^{2} \big\langle \nabla^0_{e_i} e_i, a \big\rangle \\ &= \sum_{i=1}^{2} \sigma(e_i, e_i) \langle N, a \rangle = 2H \langle N, a \rangle. \end{aligned}$$

Equation (2.23) derives from (2.22) and the properties of the Laplacian. Indeed, let $\{a_1, a_2, a_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Let $x_i = \langle x, a_i \rangle$ and $N_i = \langle N, a_i \rangle$. As $|\nabla x_i|^2 = 1 - N_i^2$, we have

$$\begin{aligned} \Delta |x|^2 &= \sum_{i=1}^{3} x_i \Delta x_i + 2 \sum_{i=1}^{3} |\nabla x_i|^2 = 4H \sum_{i=1}^{3} x_i N_i + 2 \sum_{i=1}^{3} \big(1 - N_i^2\big) \\ &= 4H \langle N, x \rangle + 4. \end{aligned}$$

□

Now assume that the mean curvature H is constant.

Proposition 2.3.2 *Let $x : M \to \mathbb{R}^3$ be an immersion with constant mean curvature. Then*

$$\Delta \langle N, a \rangle + |\sigma|^2 \langle N, a \rangle = 0. \tag{2.24}$$

$$\Delta \langle N, x \rangle = -2H - |\sigma|^2 \langle N, x \rangle. \tag{2.25}$$

Proof Fix $p \in M$ and let $\{e_1, e_2\}$ be as above. Using (2.19) and (2.20) and because $2H = \operatorname{trace}(A)$, we know that

$$2H = \sum_{i=1}^{2} \langle Ae_i, e_i \rangle = \left\langle \sum_{i=1}^{2} \nabla^0_{e_i} N, e_i \right\rangle$$

is constant on M. Thus for $j \in \{1, 2\}$ we find that

$$\begin{aligned} 0 = e_j \left(\left\langle \sum_{i=1}^{2} \nabla^0_{e_i} N, e_i \right\rangle \right) &= -\sum_{i=1}^{2} \langle \nabla^0_{e_j} \nabla^0_{e_i} N, e_i \rangle + \sum_{i=1}^{2} \langle \nabla^0_{e_i} N, \nabla^0_{e_j} e_i \rangle \\ &= \sum_{i=1}^{2} \langle \nabla^0_{e_i} \nabla^0_{e_j} N, e_i \rangle = \sum_{i=1}^{2} e_i \langle \nabla^0_{e_j} N, e_i \rangle. \end{aligned} \tag{2.26}$$

Now we compute the Laplacian of $\langle N, a \rangle$ using (2.21):

$$\begin{aligned} \Delta \langle N, a \rangle &= \sum_{i=1}^{2} e_i \langle \nabla^0_{e_i} N, a \rangle = \sum_{i,j=1}^{2} e_i \big(\langle \nabla^0_{e_i} N, e_j \rangle \langle a, e_j \rangle \big) \\ &\overset{*}{=} \sum_{i,j=1}^{2} e_i \big(\langle \nabla^0_{e_j} N, e_i \rangle \langle a, e_j \rangle \big) \\ &= \sum_{i,j=1}^{2} e_i \big(\langle \nabla^0_{e_j} N, e_i \rangle \big) \langle a, e_j \rangle + \sum_{i,j=1}^{2} \big(\langle \nabla^0_{e_j} N, e_i \rangle e_i \langle a, e_j \rangle \big) \\ &\overset{**}{=} \sum_{i,j=1}^{2} \big(\langle \nabla^0_{e_j} N, e_i \rangle e_i \langle a, e_j \rangle \big) = \sum_{i,j=1}^{2} \big(\langle \nabla^0_{e_j} N, e_i \rangle \langle a, \nabla^0_{e_i} e_j \big) \\ &= -\sum_{i,j=1}^{2} \sigma(e_i, e_j)^2 \langle N, a \rangle = -|\sigma|^2 \langle N, a \rangle. \end{aligned}$$

We have used in (*) that $\nabla^0_v N = -Av$ is a self-adjoint endomorphism; in the step (**) we apply (2.26).

As in the proof of (2.23) and for the computation of the Laplacian of $\langle N, x \rangle$, we express $\langle N, x \rangle = \sum_{i=1}^{3} N_i x_i$ with respect to an orthonormal basis of $\mathbb{R}^3$ and use the properties of the Laplacian. □

Equations (2.22) and (2.24) generalize to the other space forms. Denote by $\bar{M}^3(c)$ the space forms $\mathbb{H}^3$, $\mathbb{R}^3$ or $\mathbb{S}^3$ if $c = -1, 0$ or 1, respectively. We view the three-dimensional sphere $\mathbb{S}^3$ as a submanifold of $\mathbb{R}^4$ and the hyperbolic space $\mathbb{H}^3$ as a submanifold of the 4-dimensional Lorentz-Minkowski space $\mathbb{L}^4$ (see Chap. 10). We have [KKMS92, Rsb93]:

Proposition 2.3.3 *Let $x : M \to \bar{M}^3(c)$ be an immersion of a surface M and $a \in \mathbb{R}^4$. Then*

$$\Delta\langle x, a\rangle = -2c\langle x, a\rangle + 2H\langle N, a\rangle. \tag{2.27}$$

If, in addition, the mean curvature H is constant, then

$$\Delta\langle N, a\rangle = 2cH\langle x, a\rangle - |\sigma|^2\langle N, a\rangle. \tag{2.28}$$

Equations (2.22), (2.23), (2.24), (2.25) involve the Laplacian operator and thus are elliptic. Moreover, the last two equations have been obtained using the property that the mean curvature is constant. This justifies the title of this section. Once we compute these Laplacians, we shall obtain some geometric results for a given cmc surface. The key ingredient is that for a linear elliptic equation, such as the equation for the Laplacian operator, there is a maximum principle. In our context of compact surfaces, if f is a smooth function on M such that $\Delta f \geq 0$ (resp. ≤ 0), then $\max_M f = \max_{\partial M} f$ (resp. $\min_M f = \min_{\partial M} f$) and if f attains a maximum (resp. minimum) at some interior point of M, then f is constant [GT01].

The first result provides an estimate of the height of a compact cmc graph, usually called the *Serrin estimate* in the literature. This will be achieved by forming an appropriate linear combination of the functions $\langle x, a\rangle$ and $\langle N, a\rangle$. We first need the following auxiliary result:

Lemma 2.3.4 *Let M be an H-surface of $\mathbb{R}^3$ and $a \in \mathbb{R}^3$. Consider the function $f = H\langle x, a\rangle + \langle N, a\rangle$. Then f is constant if and only if M is umbilical.*

Proof It is straightforward that if M is an open subset of a plane or a sphere, then f is constant. Assume now that f is constant. By contradiction, suppose $p \in M$ is not umbilical. Then there exists an orthonormal frame $\{e_1, e_2\}$ of principal directions in an open set V around p, that is, $\nabla^0_{e_i} N = -\kappa_i e_i$ on V. Then

$$0 = e_i(f) = (H - \kappa_i)\langle e_i, a\rangle, \quad i = 1, 2.$$

Since there do not exist umbilical points, $H \neq \kappa_i$, $i = 1, 2$. We deduce that $\langle e_i, a\rangle = 0$ in an open subset of $V \subset M$, $i = 1, 2$. This means that $N = a$ in V and M is part of a plane. In particular, p is an umbilical point, a contradiction. □

The proof of the next theorem has its origin in the use of Bonnet's parallel surfaces by H. Liebmann and later by J. Serrin [Ser69a]. Here we follow W.H. Meeks III in [Mee88]. Further generalizations can be found in [Lop07].

Theorem 2.3.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $H \in \mathbb{R}$. Let M be an H-graph of $\mathbb{R}^3$ of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Let $e_3 = (0, 0, 1)$ and we orient M so that $\langle N, e_3\rangle \geq 0$.*

1. *If $H > 0$, then*

$$\min_{\partial\Omega} u - \frac{1}{H} \leq u \leq \max_{\partial\Omega} u.$$

In addition, if the equality holds at a point on the left-hand side, then M is part of a sphere of radius $1/H$.

2. *If $H < 0$, then*

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u - \frac{1}{H}.$$

In addition, if the equality holds at a point on the right-hand side, then M is part of a sphere of radius $-1/H$.

3. *If $H = 0$, then*

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u.$$

In the particular case $H \neq 0$ and $u = 0$ along $\partial\Omega$, we have $|u| \leq 1/|H|$ and equality attains at a point if and only if Ω is a round disk and M is a hemisphere

Proof Instead of working with the function u, we consider the surface M as an embedding $x : M \to \mathbb{R}^3$. Recall that u is the height function $u = \langle x, e_3\rangle$. As M is a cmc surface and the boundary ∂M is an embedded analytic curve, then M extends analytically across ∂M [Mul02]. Suppose $H \neq 0$. We give the proof for the case $H > 0$ (the proof is similar if $H < 0$). By (2.22), $\Delta\langle x, e_3\rangle > 0$ and by the maximum principle, $\langle x, e_3\rangle \leq \max_{\partial M}\langle x, e_3\rangle$. This proves the right inequality of item (1). Combining (2.22) and (2.24), jointly $|\sigma|^2 \geq 2H^2$, we obtain

$$\Delta\big(H\langle x, e_3\rangle + \langle N, e_3\rangle\big) = \big(2H^2 - |\sigma|^2\big)\langle N, e_3\rangle \leq 0.$$

We apply the maximum principle: since $\langle N, e_3\rangle \geq 0$, we have

$$H\langle x, e_3\rangle + \langle N, e_3\rangle \geq \min_{\partial M}\big(H\langle x, e_3\rangle + \langle N, e_3\rangle\big) \geq H \min_{\partial M}\langle x, e_3\rangle.$$

As $H > 0$,

$$\langle x, e_3\rangle \geq \frac{-\langle N, e_3\rangle}{H} + \min_{\partial M}\langle x, e_3\rangle \geq -\frac{1}{H} + \min_{\partial M}\langle x, e_3\rangle, \tag{2.29}$$

proving the left inequality of item (1).

If at a point $p \in M$, $\langle x(p), e_3\rangle = -1/H + \min_{\partial M}\langle x, e_3\rangle$, then we conclude by (2.29) that $N(p) = a$ and $p \in int(M)$. Then the function $f = H\langle x, e_3\rangle + \langle N, e_3\rangle$ attains a minimum at an interior point and this implies that f is a constant function. Lemma 2.3.4 proves that M is umbilical and as $H \neq 0$, M is part of a sphere.

Assume $H = 0$ and $u = 0$ on ∂M. Then (2.22) yields $\Delta\langle x, e_3\rangle = 0$ and the estimates follow from the maximum principle. If $H \neq 0$, $u = 0$ on ∂M and at a point, $|u| = 1/|H|$, then M is part of a sphere with planar boundary. But the estimate is sharp if M is a hemisphere. □

Observe that for small values of H, the above estimates are not good in the following sense. Assume that $u = 0$ on $\partial\Omega$. Then the graph of $u = 0$ on Ω is a minimal surface and for values near to $H = 0$, there exist H-graphs on Ω spanning

$\partial\Omega$. By continuity, these H-graphs are close to the plane $z = 0$ of $\mathbb{R}^3$ and thus their heights are small. However, the number $1/|H|$ is very large.

When the domain is not bounded, a generalization of the estimates for H-graphs will be given in Theorem 9.1.3 and Corollary 9.1.4.

Remark 2.3.6 Similar arguments to those used in Theorem 2.3.5 will be used later in different contexts to obtain 'height estimates' provided we have elliptic equations such as (2.22) and (2.24) in order to use the maximum principle. Here we recall two settings. First, consider a hypersurface M^n in a space form $\bar{M}^{n+1}(c)$. Denote by $\kappa_1, \ldots, \kappa_n$ the principal curvatures of M and let $S_r = \sum_{i_1<\cdots<i_r} \kappa_{i_1} \cdots \kappa_{i_r}$. The r-mean curvature H_r of M is defined by

$$\binom{m}{r} H_r = S_r.$$

We remark that H_1 coincides with the mean curvature H of M, $n(n-1)H_2$ is the scalar curvature and H_n is the Gauss-Kronecker curvature of M. H. Rosenberg obtained height estimates for a compact graph M^n on a geodesic hyperplane of $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$ provided for some r, H_{r+1} is a positive constant [Rsb93].

The second setting is the class of compact surfaces in Euclidean space with positive constant Gauss curvature K. If M is a such surface, the second fundamental form σ defines a Riemannian metric on M. With respect to this metric, we compute the Laplacian of the height function and the Gauss map, obtaining elliptic equations by the positivity of K [GM00a]. Then one can obtain height estimates for a graph with constant Gauss curvature $K > 0$ on a domain of a plane or a radial graph on a domain of a sphere [GM00b, Lop03b].

The second result that we derive is the Jellet theorem [Jel53], which refers to star-shaped surfaces. A *star-shaped surface* in $\mathbb{R}^3$ is a compact embedded surface such that 3-domain $W \subset \mathbb{R}^3$ that it bounds is star-shaped in the affine sense, that is, there exists a point $p_0 \in W$ such that the segment joining p_0 with any point of W is included in W.

Theorem 2.3.7 (Jellet) *The only star-shaped surface in* $\mathbb{R}^3$ *with constant mean curvature is the round sphere.*

Proof Let M be a star-shaped cmc surface and suppose after a rigid motion that p_0 is the origin of coordinates $O = (0, 0, 0)$. The property of being star-shaped is equivalent to the property that we cannot draw a straight line from O which is tangent to M. This means that the support function based on the point O has constant sign on M. Consider the orientation N that points towards W and denote by x the position vector of M. Then the support function is $\langle N, x\rangle$ and because N is the inward orientation, we see that $\langle N, x\rangle < 0$ on M. We apply the divergence theorem in (2.23). As H is constant, we find that

$$A + H \int_M \langle N, x\rangle \, dM = 0, \tag{2.30}$$

where A is the area of M. On the other hand, the divergence theorem in (2.25) yields

$$0 = 2AH + \int_M |\sigma|^2 \langle N, x\rangle \, dM \overset{*}{\leq} 2AH + 2H^2 \int_M \langle N, x\rangle \, dM$$
$$= 2H\left(A + H\int_M \langle N, x\rangle \, dM\right) \overset{**}{=} 0,$$

where we have used in $(*)$ that $|\sigma|^2 \geq 2H^2$ and (2.30) in $(**)$. Equality implies that $|\sigma|^2 = 2H^2$ on M. Then M is umbilical and this proves the theorem. □

Jellet's theorem, formulated in 1853, may be viewed as a precursor of the Hopf theorem established in 1951 [Hop83] because the genus of a star-shaped surface is zero. Following this theme, another generalization of Theorem 2.3.7 is Alexandrov's theorem, proven a century later, which asserts that a closed embedded cmc surface is a round sphere [Ale56, Ale62]. In this case, a star-shaped surface is embedded.

A third application of the computations of the above Laplacians is the following result.

Theorem 2.3.8 *Planar disks and small spherical caps are the only cmc graphs spanning a circle.*

As we shall see in the next chapter, this result is trivial by the tangency principle (see Theorem 3.2.6). However, the purpose here is to give a proof that does not involve the tangency principle but uses Eqs. (2.22) and (2.24) [Lop09]. In a similar context and for a closed cmc surface, R. Reilly obtained another proof of Alexandrov's theorem without the use of the maximum principle thanks to a combination of the Minkowski formulae [Rei82].

Proof Let $\Gamma \subset \mathbb{R}^2 \times \{0\}$ be a circle of radius $r > 0$ about the origin and denote by Ω the domain bounded by Γ. Consider a compact H-surface which is a graph of u defined on Ω and with $u = 0$ on $\partial\Omega$. Then u satisfies (1.5) where H is computed by the unit normal vector given in (2.2). Then $\langle N, e_3\rangle > 0$ on M, where $e_3 = (0, 0, 1)$. Let α be the parametrization of Γ such that $\alpha' \times \nu = N$, where ν is the inner unit conormal vector of M along Γ. We know that $\alpha'' = -\alpha/r^2$. Since $\langle N, e_3\rangle > 0$ on M, we see that $\alpha \times \alpha' = re_3$ and

$$\langle \nu, e_3\rangle = \langle N \times \alpha', e_3\rangle = \langle N, \alpha' \times e_3\rangle = \frac{1}{r}\langle N, \alpha\rangle. \tag{2.31}$$

Using (1.5), we obtain

$$-2\pi r^2 H = \int_{\partial\Omega} \left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \mathbf{n}\right\rangle = \int_{\partial M} \langle \nu, e_3\rangle \, ds, \tag{2.32}$$

where $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. With this equation and integrating (2.22), we have

$$\int_M \langle N, e_3\rangle \, dM = \pi r^2. \tag{2.33}$$

Integrating (2.24), we obtain

$$\int_M |\sigma|^2 \langle N, e_3\rangle \, dM = \int_{\partial M} \langle dN\nu, e_3\rangle \, ds. \tag{2.34}$$

We study each side of (2.34). In view of (2.33), since $|\sigma|^2 \geq 2H^2$ and $\langle N, e_3\rangle > 0$ on M, the left-hand side of (2.34) yields

$$\int_M |\sigma|^2 \langle N, e_3\rangle \, dM \geq 2H^2 \int_M \langle N, e_3\rangle \, dM = 2\pi r^2 H^2. \tag{2.35}$$

Now we turn the attention to the right-hand side of (2.34). First, note that

$$dN\nu = -\sigma\big(\alpha', \nu\big)\alpha' - \sigma(\nu, \nu)\nu.$$

From (2.31), we see that

$$\begin{aligned}\sigma(\nu, \nu) &= 2H - \sigma\big(\alpha', \alpha'\big) = 2H + \big\langle dN\alpha', \alpha'\big\rangle \\ &= 2H - \big\langle N, \alpha''\big\rangle = 2H + \frac{1}{r^2}\langle N, \alpha\rangle = 2H + \frac{1}{r}\langle \nu, e_3\rangle.\end{aligned} \tag{2.36}$$

By the choices of α' and N, an integration of the constant mean curvature equation (1.5) gives

$$2\pi r^2 H = \int_{\partial\Omega} \left\langle \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \mathbf{n}\right\rangle ds = -\int_{\partial M} \langle \nu, e_3\rangle \, ds.$$

Since $\langle \alpha', e_3\rangle = 0$ and using this relation together with (2.36), we find that

$$\begin{aligned}\int_{\partial M} \langle dN\nu, e_3\rangle \, ds &= -\int_{\partial M} \sigma(\nu, \nu)\langle \nu, e_3\rangle \, ds = -\int_{\partial M} \left(2H + \frac{1}{r}\langle \nu, e_3\rangle\right)\langle \nu, e_3\rangle \, ds \\ &= 4\pi r^2 H^2 - \frac{1}{r}\int_{\partial M} \langle \nu, e_3\rangle^2 \, ds.\end{aligned} \tag{2.37}$$

We employ the Cauchy-Schwarz inequality in (2.32), obtaining

$$\int_{\partial M} \langle \nu, e_3\rangle^2 \, ds \geq \frac{1}{2\pi r}\left(\int_{\partial M} \langle \nu, e_3\rangle \, ds\right)^2 = 2\pi r^3 H^2. \tag{2.38}$$

Equation (2.37) and inequality (2.38) imply

$$\int_{\partial M} \langle dN\nu, e_3\rangle \, ds \leq 2\pi r^2 H^2. \tag{2.39}$$

By combining (2.34), (2.35) and 2.39, we obtain

$$2\pi r^2 H^2 \leq \int_M |\sigma|^2 \langle N, e_3\rangle \, dM = \int_{\partial M} \langle dN\nu, e_3\rangle \, ds \leq 2\pi r^2 H^2.$$

Therefore, we have equalities in all the inequalities, in particular, $|\sigma|^2 = 2H^2$ on M. This means that M is an umbilical surface, in particular, it is an open subset of a plane or a sphere. Since the boundary of M is a circle and M is a graph, then M is a planar disk or a small spherical cap. □

We end the chapter applying the Laplacians of x and N to the problem of the stability of a cmc surface. First, we show that a cmc graph is always strongly stable. We need the following lemma [FS80, Theorem 1]:

Lemma 2.3.9 *Let M be a compact cmc surface and assume that there exists a function g on M such that $g \neq 0$ on M and $\Delta g + |\sigma|^2 g = 0$. Then M is strongly stable and $\lambda_1(L) > 0$.*

Proof Let $f \in C^\infty(M)$ with $f = 0$ on ∂M. Assume that $g > 0$ on M and define $h = \log(g)$. Then $\Delta h = -|\sigma|^2 - |\nabla h|^2$. Multiplying by f^2 and integrating on M, we have

$$\int_M |\sigma|^2 f^2 \, dM + \int_M f^2 |\nabla h|^2 \, dM = -\int_M f^2 \Delta h \, dM. \tag{2.40}$$

As $\operatorname{div}(f^2 \nabla h) = f^2 \Delta h + 2f\langle \nabla f, \nabla h\rangle$, the divergence theorem yields

$$\begin{aligned} -\int_M f^2 \Delta h \, dM &= 2\int_M f\langle \nabla f, \nabla h\rangle \, dM \leq 2\int_M |f||\nabla h||\nabla f| \\ &\leq \int_M f^2 |\nabla h|^2 \, dM + \int_M |\nabla f|^2 \, dM. \end{aligned}$$

Combining this inequality with (2.40), we get $I(f) \geq 0$. In fact, if $f \neq 0$, $I(f) > 0$ because if $I(f) = 0$, f is proportional to h, contradicting that $h \neq 0$ along ∂M. □

Theorem 2.3.10 *A cmc graph is strongly stable.*

Proof Assume that the surface is a graph on a plane and let a be a unit vector orthogonal to this plane. The function $g = \langle N, a\rangle$ does not vanish and satisfies (2.24). Then we apply Lemma 2.3.9. □

We have seen in Proposition 2.1.8 that a cmc graph minimizes the area but, in fact, this property extends to strong stable surfaces, more precisely, to surfaces with $\lambda_1(L) > 0$. Without entering into details, we explain the main arguments. Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface with constant mean curvature H_0.

For each $u \in C_0^{2,\alpha}(M)$, with sufficiently small $C^{2,\alpha}$-norm, consider the immersion $x_t = x + tuN$, $t \in (-\varepsilon, \varepsilon)$ and denote by $H(u,t)$ the mean curvature of x_t. Define

$$F : C_0^{2,\alpha}(M) \times (-\varepsilon, \varepsilon) \to C_0^{\alpha}(M), \qquad F(u,t) = H(u,t) - H_0.$$

Then $F(0,0) = 0$. By a straightforward computation, the derivative of F with respect to the first variable is $(D_1 F)_{(0,0)}(v) = \Delta v + |\sigma|^2 v$, that is, the Jacobi operator of M (see also Lemma 8.1.4). Because $\lambda_1(L) \neq 0$, the Fredholm alternative proves that $D_1 F$ is an isomorphism (Lemma 8.1.3). The implicit function theorem for Banach spaces implies the existence of $\varepsilon' < \varepsilon$, an open set V around $0 \in C_0^{2,\alpha}(M)$ and a smooth function $\phi : (-\varepsilon', \varepsilon') \to V$, $\phi(0) = 0$, such that $F(\phi(t), t) = 0$ for all $t \in (-\varepsilon', \varepsilon')$. Then the variation $x_t = x + t\phi(t)N$ is formed by immersions of constant mean curvature H_0 and with the same boundary as M. See also [Tom75] for deformations of cmc disks with the same mean curvature and [Koi02] for variations of cmc surfaces but with distinct values of H. In the particular case that x is an embedding, the immersions x_t are also embedded. Let $\mathcal{U} = \cup\{x_t(M) : t \in (-\varepsilon', \varepsilon')\}$.

Let M' be a compact embedded surface with the same volume as M and $\partial M' = \partial M$. Suppose that M' is sufficiently close to M in the $C^{2,\alpha}$-norm in order to ensure that $M' \subset \mathcal{U}$. As $\mathrm{vol}(M) = \mathrm{vol}(M')$, the two surfaces M and M' enclose a domain $W \subset \mathbb{R}^3$ with zero volume. Since $W \subset \mathcal{U}$, each point of W belongs to an immersion x_t. Define on W the vector field Y that assigns to each point $x_t(p)$ the unit normal vector field at x_t. As all surfaces $x_t(M)$ are H_0-surfaces, $\mathrm{Div}(Y) = 2H_0$ on $\mathcal{U}$. If N and N' denote the unit normal vector fields on M and M', respectively, then $\langle Y, N\rangle = 1$ and $\langle Y, N'\rangle \leq 1$. A similar argument as in Proposition 2.1.8 concludes finally [Gro96]:

Theorem 2.3.11 *Let M be a compact embedded cmc surface. If $\lambda_1(L) > 0$, any embedded surface M' sufficiently close to M and with the same volume satisfies* $\mathrm{area}(M) \leq \mathrm{area}(M')$.

An argument using the tangency principle (Theorem 3.2.4) proves that, indeed, $\mathrm{area}(M) = \mathrm{area}(M')$ if and only if $M' = M$.

If the surface is closed, we have the Barbosa-do Carmo theorem [BC84]; see also [Wen91].

Theorem 2.3.12 (Barbosa-do Carmo) *The only closed stable cmc surface in Euclidean space is the round sphere.*

Proof By Eq. (2.23) and the divergence theorem, the function $f = 1 + H\langle N, x\rangle$ satisfies

$$\int_M f \, dM = 0.$$

In particular, f is a test function for the stability operator (2.11). Let $h = \langle N, x\rangle$. Equation (2.25) gives

$$\int_M \big(2H + |\sigma|^2 h\big)\, dM = 0. \tag{2.41}$$

We compute $I(f)$ using (2.41):

$$\begin{aligned} I(f) &= -\int_M f\big(\Delta f + |\sigma|^2 f\big)\, dM \\ &= -\int_M (1+Hh)\big(-2H^2 - |\sigma|^2 Hh + |\sigma|^2(1+Hh)\big)\, dM \\ &= \int_M \big(2H^2 - |\sigma|^2\big)\, dM \leq 0, \end{aligned}$$

where in the last inequality we have employed $2H^2 \leq |\sigma|^2$. Since M is stable, $I(f) \geq 0$ and then we deduce $I(f) = 0$. Hence we have $2H^2 = |\sigma|^2$ on M, i.e., M is an umbilical (closed) surface and the immersion x describes a round sphere. □

Chapter 3
The Comparison and Tangency Principles

Consider a surface given as the graph of a function $z = u(x, y)$. Equation (1.5) expresses the mean curvature H of the surface in terms of the function u and its first and second derivatives. Then a geometric condition such as constant mean curvature can be treated from the analytical viewpoint by studying a partial differential equation. This allows the use of the machinery of calculus to obtain geometric results on the shape of the initial surface. In this chapter we study what can be deduced when two surfaces with the same constant mean curvature are tangent at a common point. In terms of the PDEs theory, our scenario consists of two functions that satisfy a second order partial differential equation where both functions agree up to the first order at one point.

We begin this chapter with the comparison principle, which it is a basic application of calculus comparing the values of the mean curvatures of two surfaces tangent at a point. Next we introduce the tangency principle which is, in our context, the geometric version of the maximum principle for elliptic linear equations of second order. Using the tangency principle, we shall obtain a set of results concerning the shape of a compact cmc surface in terms of the geometry of its boundary. In particular, we will study cmc surfaces with planar boundary and we shall compare the given surface with known surfaces with the same mean curvature, such as pieces of spheres and cylinders.

3.1 The Comparison Principle

We identify $\mathbb{R}^2$ with the horizontal Euclidean plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and put

$$e_3 = (0, 0, 1).$$

We take (x, y, z) to be the usual coordinates of $\mathbb{R}^3$. Let M be the graph $z = u(x, y)$ of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ defined on a domain $\Omega \subset \mathbb{R}^2$. We orient M with

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_3,
© Springer-Verlag Berlin Heidelberg 2013

the unit normal vector field N given by

$$N\big(x, y, u(x, y)\big) = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + u_x^2 + u_y^2}}(x, y) = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}(x, y). \tag{3.1}$$

Since $\langle N, e_3\rangle > 0$, we say that N points upwards (for the reverse orientation $-N$, we say that it points downwards). A computation of the mean curvature H of M gives

$$\big(1 + u_y^2\big)u_{xx} - 2u_x u_y u_{xy} + \big(1 + u_x^2\big)u_{yy} = 2H\big(1 + |\nabla u|^2\big)^{\frac{3}{2}}, \tag{3.2}$$

which is written in divergence form as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 2H. \tag{3.3}$$

The expression (3.2) is called in the literature the *constant mean curvature equation*.

Remark 3.1.1 Sometimes we will change Eq. (3.2) using the downward orientation. In this case a minus sign will appear on the right-hand side of (3.2) (also in (3.3)). This will allow us to associate the idea of 'positive values' of H with surfaces that lie 'above' the plane $z = 0$.

In the theory of partial differential equations, Eq. (3.2) falls into the class of elliptic equations because the symmetric matrix of coefficients of second order in (3.2), which is given by

$$\begin{pmatrix} 1 + u_y^2 & -u_x u_y \\ -u_x u_y & 1 + u_x^2 \end{pmatrix},$$

is positive definite. For an elliptic linear equation, the maximum principle plays a special role which will be treated in the next section. First we focus on an easy application of calculus for the mean curvature equation. Equation (3.2) looks like the Laplace equation $\Delta u = 0$. In fact, Eq. (3.2) can be expressed as

$$\big(1 + |\nabla u|^2\big)\Delta u - \big(\nabla^2 u\big)(\nabla u, \nabla u) = 2H\big(1 + |\nabla u|^2\big)^{\frac{3}{2}}$$

where $\nabla^2 u$ is the Hessian of u. Given a point $q \in \Omega$, it is possible to make a change of coordinates around $q \in \Omega$ so that the above equation reduces to $\Delta u = 2H$ at q.

Lemma 3.1.2 *Let $p \in M$. There exist local coordinates around p such that M is $z = u(x, y)$ and $\Delta u = 2H$ at p.*

Proof Let M be the graph of $z = u(x, y)$ around p, where u is defined on a domain $\Omega \subset \mathbb{R}^2$, $q \in \Omega$ and $p = (q, u(q))$. After a rigid motion of $\mathbb{R}^3$ if necessary, assume that the tangent plane $T_p M$ is horizontal. Then $\nabla u(q) = (0, 0)$ and the expression (3.2) is finally $\Delta u(q) = 2H(p)$. □

In order to state the comparison principle, we require some definitions.

Definition 3.1.3 Let $\Omega \subset \mathbb{R}^2$ be a smooth domain. Consider two graphs M_1 and M_2 on $\overline{\Omega}$ of functions u_1 and u_2 respectively. Assume that M_1 and M_2 are tangent at a common point $p = (q, u_i(q))$, $q \in \overline{\Omega}$. We orient M_1 and M_2 by Gauss maps N_1 and N_2 respectively such that $N_1(p) = N_2(p) = e_3$. We say that M_1 lies above M_2 around p if $u_1 \geq u_2$ in a neighborhood of q.

As an immersed surface is locally the graph of a function, we extend the above definition to immersed surfaces.

Definition 3.1.4 Let M_1 and M_2 be two immersed surfaces in $\mathbb{R}^3$ that are tangent at some common point p, which satisfies $p \in int(M_1) \cap int(M_2)$ or $p \in \partial M_1 \cap \partial M_2$. In the last case we assume, in addition, $T_p \partial M_1 = T_p \partial M_2$. After a rigid motion of $\mathbb{R}^3$, we assume that $N_1(p) = N_2(p) = e_3$. We say that M_1 lies above M_2 around p if this holds locally around p.

The definition needs some further clarification.

1. Suppose that M_i is locally the graph of a function u_i defined in a domain Ω_i of the common tangent plane $T_p M_i$, $i = 1, 2$. We write $p = (q, u_i(q))$ and $\tilde{M}_i = \text{graph}(u_i) \subset M_i$. If p is an interior point, it is clear that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and there exists a subdomain Ω with $q \in \Omega \subset \Omega_1 \cap \Omega_2$.
2. If p is a boundary point, we are assuming that $\tilde{M}_1 \geq \tilde{M}_2$ around p, in particular, u_1 and u_2 are comparable around q. Moreover, the condition $T_p \partial \tilde{M}_1 = T_p \partial \tilde{M}_2$ guarantees that there exists a smooth subdomain $\Omega \subset \Omega_1 \cap \Omega_2$ where $q \in \partial \Omega$.
3. In general, two surfaces M_1 and M_2 tangent at a boundary point may not be graphs in the same domain, even assuming $T_p \partial M_1 = T_p \partial M_2$. For example, take two disjoint domains $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ such that $\overline{\Omega_1} \cap \overline{\Omega_2} \neq \emptyset$. If $p \in \partial \Omega_1 \cap \partial \Omega_2$, then there are graphs on $\overline{\Omega_i}$ tangent at p which are not graphs in a common domain.
4. If a surface M is embedded, then locally around a point $p \in M$, M is a graph, and this graph is uniquely determined around p. If the surface is immersed, say, $x : M \to \mathbb{R}^3$, a point $p \in x(M)$ could have many preimages. When we say $M_1 \geq M_2$ around p we refer to a preimage of p at each surface M_i.

The following result is a direct application of the calculus of two variables and we do not assume that the mean curvature is constant.

Theorem 3.1.5 (Comparison principle) *Let M_1 and M_2 be two surfaces with a common tangent point p. If $M_1 \geq M_2$ around p, then $H_1(p) \geq H_2(p)$.*

Proof Since the result is local, we work with graphs and we use the above notation. After a change of coordinates, assume that the tangent plane $T_p M_i$ is $\mathbb{R}^2$ and

$N_i(p) = e_3$. This means that $\nabla u_i(x_0, y_0) = (0, 0)$, $i = 1, 2$. By (3.2), the mean curvatures H_i are

$$2H_i(p) = \Delta u_i(x_0, y_0).$$

Since $M_1 \geq M_2$ around p, the difference function $w = u_1 - u_2$ has a local minimum at (x_0, y_0). It follows that $\nabla^2 w$ is positive semi-definite at (x_0, y_0). In particular,

$$0 \leq w_{xx}(x_0, y_0) = (u_1)_{xx}(x_0, y_0) - (u_2)_{xx}(x_0, y_0)$$
$$0 \leq w_{yy}(x_0, y_0) = (u_1)_{yy}(x_0, y_0) - (u_2)_{yy}(x_0, y_0).$$

This implies

$$0 \leq \Delta w(x_0, y_0) = \Delta u_1(x_0, y_0) - \Delta u_2(x_0, y_0) = 2\big(H_1(p) - H_2(p)\big). \qquad \square$$

We illustrate some applications of the comparison principle by means of a set of results concerning cmc surfaces with boundary. In the first result constant mean curvature is not needed.

Corollary 3.1.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$. Let $M = graph(u)$. If $H \neq 0$ on M, then $u > 0$ on Ω or $u < 0$ on Ω.*

Proof We argue by contradiction. Suppose that M has interior points on both sides of $\mathbb{R}^2$. Let p_0, $p_1 \in int(M)$ be the points of highest and lowest height with respect to $\mathbb{R}^2$, respectively. Both points exist by the compactness of M. Let $p_0 = (x_0, y_0, z_0)$ and $p_1 = (x_1, y_1, z_1)$ and assume that $z_1 \leq 0 < z_0$ (similarly if $z_1 < 0 \leq z_0$). Since p_0 and p_1 are interior points, the tangent planes of M at both points are horizontal. Fix the orientation N on M pointing downwards.

Consider the (horizontal) tangent plane $T_{p_0}M$ placed at p_0 with the same orientation at p_0 as M. Then $M \geq T_{p_0}M$ around p_0 and the comparison principle implies $H(p_0) \geq 0$ because $T_{p_0}M$ is a minimal surface (see Fig. 3.1). On the other hand, at the point p_1 we place the horizontal plane $T_{p_1}M$. Reversing the orientation on $T_{p_1}M$ if necessary, $T_{p_1}M \geq M$ around p_1. Then $0 \geq H(p_1)$. By connectedness, there would be a point of M where H vanishes, which leads to a contradiction. $\square$

We may also argue as follows. As $T_{p_0}M$ is a horizontal plane, then Eq. (3.2) reduces to $\Delta u(x_0, y_0) = -2H(p_0)$. Similarly, $\Delta u(x_1, y_1) = -2H(p_1)$. Recall the orientation points downwards and so, the right-hand side of (3.2) has a minus sign. As u attains a maximum at (x_0, y_0) and a minimum at (x_1, y_1), $\Delta u(x_0, y_0) \leq 0 \leq \Delta u(x_1, y_1)$. Thus $H(p_0) \geq 0 \geq H(p_1)$ as above, obtaining the same contradiction.

From the proof, we deduce (see also Theorem 2.3.5 when H is constant):

Corollary 3.1.7 *Let M be a graph on a bounded domain $\Omega \subset \mathbb{R}^2$ with $\partial M = \partial\Omega$. Fix on M the orientation pointing downwards. Assume that $H \neq 0$ on M. If $H > 0$ (resp. $H < 0$) then M lies above (resp. below) the plane $z = 0$.*

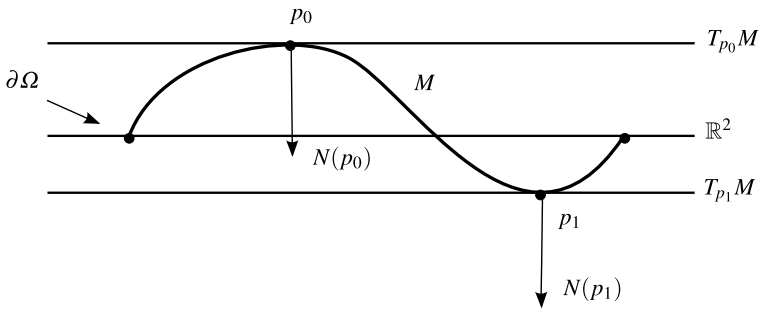

Fig. 3.1 Proof of Corollary 3.1.6. Comparison of the graph M with the tangent planes at the highest and lowest points of M

Corollary 3.1.8 (Monotonicity) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and consider M_i, $i = 1, 2$, two graphs of $u_i \in C^2(\Omega) \cap C^0(\overline{\Omega})$, respectively, with $u_1 = u_2$ in $\partial\Omega$. If the corresponding mean curvature functions H_1 and H_2 with respect to the downward orientation satisfy $H_1 < H_2$ on Ω, then $u_1 < u_2$ in Ω.*

Let us observe that if we take M_1 to be the plane $z = 0$, then we obtain Corollary 3.1.7.

Proof By the comparison principle, it is not possible that at a point $q \in \Omega$, $u_1(q) = u_2(q)$ and $u_1 \leq u_2$ around q: in such a case, $M_1 \geq M_2$ around $(q, u_i(q))$ and the comparison principle yields $H_1(q) \geq H_2(q)$.

Let us suppose by contradiction that at a point $q_0 \in \Omega$, $u_1(q_0) > u_2(q_0)$. Then the difference function $w = u_1 - u_2$ attains a maximum at a point q_1. As $w(q_1) > 0$ and $w = 0$ along $\partial\Omega$, then $q_1 \in \Omega$. Then $\nabla w(q_1) = (0, 0)$ and $\Delta w(q_1) \leq 0$. Now Eq. (3.2) gives

$$\Delta w(q_1) = -2(H_1 - H_2)(q_1) \leq 0,$$

obtaining a contradiction. □

Corollary 3.1.9 *Let $H \neq 0$ and let $\Omega \subset \mathbb{R}^2$ be a domain. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $M = graph(u)$ is an H-surface with respect to the downward orientation. If $H > 0$ (resp. $H < 0$), then u does not have an interior local minimum (resp. interior local maximum).*

Proof Assume $H > 0$. If there is an interior local minimum p, we place the tangent plane T_pM at p. By the orientation given on M, and choosing on T_pM the downward orientation, $T_pM \geq M$ around p. The comparison principle yields $0 \geq H$, a contradiction. The case $H < 0$ is similar. □

Corollary 3.1.10 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let M be a compact embedded surface in $\mathbb{R}^3$ with mean curvature function $H \neq 0$ with $\partial M = \partial\Omega$. If $M \subset \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$, then $int(M) \cap \Omega = \emptyset$.*

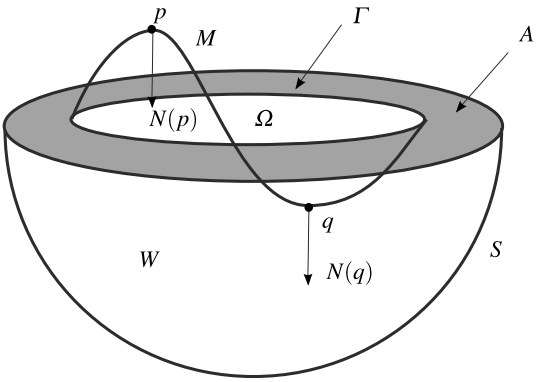

Fig. 3.2 Proof of Theorem 3.1.11. The surface M, the hemisphere S and the annulus $A \subset P$ enclose a 3-domain W. If N points towards W, at the highest point of M, $N(p)$ points downwards and at the lowest point, $N(q)$ points downwards again

Proof Assume, by contradiction, that $p \in int(M) \cap \Omega$. Since M lies above $\mathbb{R}^2$, there exists a subdomain $\Omega' \subset \Omega$ such that $M \cup \Omega'$ defines a bounded 3-domain W. We orient M by N so that N points towards W. If $p_1 \in M$ is the highest point of M, then $N(p_1) = -e_3$. Thus $M \geq T_{p_1}M$ around p_1 and the comparison principle gives $H(p_1) \geq 0$. As $H \neq 0$ on M, we have $H > 0$. As p is an interior point, the tangent plane at p is horizontal and thus, $N(p) = -e_3$. As a consequence, if we orient P downwards, then $P \geq M$ around p. The comparison principle implies $0 \geq H(p)$, a contradiction. □

The following theorem is due to M. Koiso [Koi86] and, again, it does not require the assumption of constant mean curvature.

Theorem 3.1.11 *Let Γ be a simple closed curve included in a plane P and denote by $\Omega \subset P$ the domain bounded by Γ. Assume that M is a compact embedded surface in $\mathbb{R}^3$ spanning Γ with $H \neq 0$. If $M \cap (P \setminus \overline{\Omega}) = \emptyset$, then $int(M)$ lies on one side of P.*

We say that a set lies on one side of a plane P if it is included in one of the two connected components of $\mathbb{R}^3 \setminus P$.

Proof We proceed by contradiction. Suppose that P is the plane $z = 0$. Consider a hemisphere S below P with $\partial S \subset P$ and denote by $D \subset P$ the closed disk bounded by ∂S. Take S of radius sufficiently large such that $\overline{\Omega} \subset D$ and the part of M below P is contained in the domain enclosed by $S \cup D$. If A denotes the annulus $D \setminus \overline{\Omega}$, then $M \cup A \cup S$ defines a non-smooth closed embedded surface in $\mathbb{R}^3$. By the Alexander duality [GH81], this surface bounds a 3-domain $W \subset \mathbb{R}^3$. See Fig. 3.2. We orient M by the unit normal vector N pointing to W.

Denote by p and q the highest and lowest points of M with respect to P, respectively, with $p, q \in int(M)$. Suppose $z(q) \leq 0 < z(p)$ (the argument is similar if $z(q) < 0 \leq z(p)$). Observe that if $z(q) = 0$, then $q \in \Omega$ and the result follows from Corollary 3.1.10. Since $N(p)$ points to W, $N(p) = -e_3$. Compare M with the (horizontal) tangent plane T_pM at p, where T_pM is oriented pointing downwards. Since

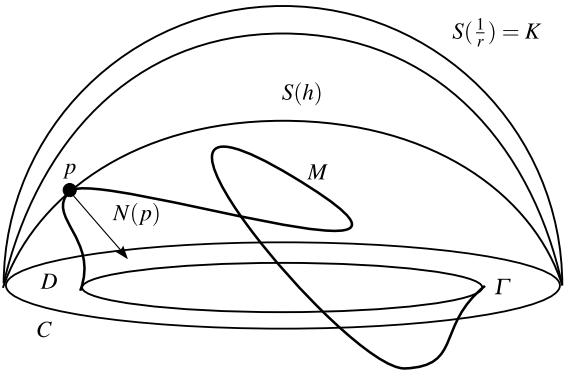

Fig. 3.3 Proof of Theorem 3.1.12, comparing the surface M with the spherical cap $S(h)$

T_pM is a minimal surface and $M \geq T_pM$ around p, the comparison principle implies $H(p) \geq 0$. As $H \neq 0$, then $H > 0$ on M. Now, we work with the point $q \in M$. Since $N(q)$ points to W, we have $N(q) = -e_3$. Compare the horizontal plane T_qM with M at q. Let us take on T_qM the orientation pointing downwards. In this situation, $T_qM \geq M$ around q and the comparison principle now yields $H(q) \leq 0$, a contradiction. □

Theorem 3.1.11 may be viewed as a generalization of Corollary 3.1.6. If we compare both results, the fact that the surface was a graph allowed us in Corollary 3.1.6 to fix a suitable orientation on the surface. In Theorem 3.1.11 the orientation is determined by the assumption $M \cap (P \setminus \overline{\Omega}) = \emptyset$.

We point out that we may replace Γ, under appropriate modifications, by a disconnected plane curve. So, assume that $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$ is the decomposition into its connected components and $\Omega_i \subset P$ is the domain bounded by Γ_i. Then it suffices to assume that $M \cap ext(\Gamma) = \emptyset$, where $ext(\Gamma) = \bigcap_{i=1}^{n}(P \setminus \overline{\Omega_i})$.

Finally, we treat the minimal case, that is, $H = 0$. We drop the hypothesis that the surface is a graph and we assume that the surface is immersed.

Theorem 3.1.12 *Let P be a plane in $\mathbb{R}^3$. Let M be a compact surface and let $x : M \to \mathbb{R}^3$ be a minimal immersion. If $x(\partial M) \subset P$, then $x(M) \subset P$.*

The compactness of M is necessary since the half catenoid $z = \operatorname{arc\,cosh}(\sqrt{x^2 + y^2})$, $x^2 + y^2 \geq 1$, has planar boundary.

Proof The reasoning is by contradiction. Without loss of generality, assume that P is the plane $z = 0$ and that M has points above P. Let $C \subset P$ be a circle of radius $r > 0$ and let K denote the hemisphere with $\partial K = C$ that lies above P. The circle C bounds a domain $D \subset P$ and $D \cup K$ defines a bounded 3-domain W. Take r sufficiently large so that $\Gamma \subset int(D)$ and $x(M) \cap \{(x, y, z) \in \mathbb{R}^3 : z > 0\} \subset W$ as depicted in Fig. 3.3.

Let $S(H)$ be the family of small spherical caps above P with $\partial S(H) = C$ and parameterized by their mean curvatures $H > 0$ with the downward orientation. Then

$0 < H \leq 1/r$. For $H = 1/r$, $S(1/r)$ is the hemisphere K. Starting from the value $H = 1/r$, let $H \searrow 0$ until the first time $H = h$ that $S(H)$ touches $x(M)$. This occurs necessarily at points of $x(M)$ strictly above the plane P. Let $p \in S(h) \cap M$. In particular, $S(h)$ and M are tangent at p. We orient M so that its orientation agrees with $S(h)$ at p. Then $M \geq S(h)$ around p and, as M is minimal, $0 \geq h$ by the comparison principle; this is a contradiction. □

This theorem establishes that in the simplest case that the boundary is planar, a minimal surface must be part of a plane. As a consequence, if we want to investigate H-surfaces spanning a plane closed curve, the interesting case is $H \neq 0$.

Corollary 3.1.13 *There do not exist closed minimal surfaces.*

Proof If M is such a minimal closed surface, near to the farthest point $p \in M$ from the origin, the intersection of M by a parallel plane to T_pM gives a minimal graph bounded by a closed curve but the surface is not planar, a contradiction. □

This last result can also be proved by comparing M with spheres. If M is a closed minimal surface, we consider a large ball $B \subset \mathbb{R}^3$ including M. Then $S = \partial B$ is a sphere, which is a surface with positive mean curvature if we orient S with the inward orientation. By moving S until it first intersects M at a point p, at this position M and S are tangent because p is interior in both surfaces. As $M \geq S$ around p, the comparison principle yields a contradiction.

3.2 The Tangency Principle

In the preceding section we have compared the mean curvatures of two surfaces that are tangent at a point p and where one of the surfaces lies above the other one around p. The interest now is what happens if such surfaces have the same constant mean curvature. In such case, and from a local viewpoint, we have two graphs of functions u_i, $i = 1, 2$, with $u_1(q) = u_2(q)$, $u_1 \leq u_2$ around q, and u_i satisfy the same Eq. (3.2).

For a *linear* elliptic equation of second order, the Dirichlet problem has a unique solution, which is a direct consequence of the maximum principle. Although Eq. (3.2) is not linear, the key point is that the difference of two solutions of this equation satisfies a linear equation, and hence it satisfies the hypothesis of the maximum principle. We shall make use of this in this section. The topic of maximum principles for elliptic equations is extensive, but we will only use those statements necessary for our needs. We refer to [GT01, PW67] as general guides.

Let $\Omega \subset \mathbb{R}^2$ be a smooth domain and consider a linear differential operator of second order $L : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ given by

$$Lu = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y, \tag{3.4}$$

where $a_{ij}, b_i \in C^\infty(\overline{\Omega})$. We say that L is elliptic if the symmetric matrix $A(x, y) = (a_{ij}(x, y))$ is positive definite for all $(x, y) \in \overline{\Omega}$, where $a_{21} = a_{12}$. As the matrix $A(x, y)$ is symmetric, it is diagonalizable. Denote by $\lambda(x, y) \leq \Lambda(x, y)$ the two eigenvalues of $A(x, y)$ for each $(x, y) \in \Omega$. We say that L is uniformly elliptic if Λ/λ is bounded in $\overline{\Omega}$. For example, it suffices that $\lambda \geq \lambda_0 > 0$ on $\overline{\Omega}$, for some constant λ_0.

We state the classical strong maximum principle of E. Hopf [Hop27]

Theorem 3.2.1 (Hopf maximum principle, interior point version) *Let L be a uniformly elliptic linear operator. Let $u \in C^\infty(\overline{\Omega})$ satisfy $Lu \geq 0$. If u attains its maximum at $p_0 \in \Omega$, then u is constant.*

Theorem 3.2.2 (Hopf maximum principle, boundary point version) *Let L be as above and $u \in C^\infty(\overline{\Omega})$ such that $Lu \geq 0$. If u attains its maximum at $p_0 \in \partial\Omega$ and $\partial u/\partial \mathbf{n}(p_0) \leq 0$, where $\mathbf{n}$ indicates the outward unit conormal vector of Ω along $\partial\Omega$ at p_0, then u is constant.*

We remark that the Hopf maximum principle is still valid for other types of domains, such as, for example, if $\Omega \subset \mathbb{R}^2$ satisfies the uniform interior sphere condition. This property means that for any point $p \in \partial\Omega$, there is a disk $D_r(q) \subset \Omega$ such that $\overline{D_r(q)} \cap \partial\Omega = \{p\}$. In other cases of domains, it is possible to extend the Hopf maximum principle with extra assumptions but we do not enter into this discussion. We will only need to extend the principle in one special setting, namely, when Ω is a quadrant of $\mathbb{R}^2$, but we will not address this case until we need to use it (Theorem 4.1.16).

We prove that we can apply the Hopf maximum principle to the difference of two solutions of the constant mean curvature equation, showing that it satisfies a linear elliptic equation.

Lemma 3.2.3 *Let $\Omega \subset \mathbb{R}^2$ be a domain and $u_i \in C^\infty(\overline{\Omega})$, $i = 1, 2$. Denote by H_i the mean curvature of the graph of u_i, $i = 1, 2$, computed according to (3.1). If $H_2 \geq H_1$ in $\overline{\Omega}$, then the difference function $w = u_2 - u_1$ satisfies $Lw \geq 0$, where L is a uniformly elliptic linear operator.*

Proof In view of (3.2), define $F : \mathbb{R}^5 \to \mathbb{R}$ by

$$F(x_1, x_2, x_3, x_4, x_5) = \frac{(1 + x_5^2)x_1 - 2x_4x_5x_2 + (1 + x_4^2)x_3}{(1 + x_4^2 + x_5^2)^{3/2}}.$$

For each $0 \leq \tau \leq 1$, define the following smooth functions on $\overline{\Omega}$

$$\begin{aligned} r(\tau) &= (1 - \tau)(u_1)_{xx} + \tau(u_2)_{xx}, \\ s(\tau) &= (1 - \tau)(u_1)_{xy} + \tau(u_2)_{xy}, \\ t(\tau) &= (1 - \tau)(u_1)_{yy} + \tau(u_2)_{yy}, \end{aligned}$$

$$p(\tau) = (1-\tau)(u_1)_x + \tau(u_2)_x,$$
$$q(\tau) = (1-\tau)(u_1)_y + \tau(u_2)_y.$$

The fundamental theorem of calculus gives

$$\begin{aligned}
0 &\le 2(H_2 - H_1)\\
&= F\big(r(1), s(1), t(1), p(1), q(1)\big) - F\big(r(0), s(0), t(0), p(0), q(0)\big)\\
&= \int_0^1 \frac{d}{d\tau} F\big(r(\tau), s(\tau), t(\tau), p(\tau), q(\tau)\big) d\tau\\
&= \left(\int_0^1 \frac{1+q(\tau)^2}{(1+p(\tau)^2+q(\tau)^2)^{3/2}} d\tau\right) w_{xx}\\
&\quad - 2\left(\int_0^1 \frac{p(\tau)q(\tau)}{(1+p(\tau)^2+q(\tau)^2)^{3/2}} d\tau\right) w_{xy}\\
&\quad + \left(\int_0^1 \frac{1+p(\tau)^2}{(1+p(\tau)^2+q(\tau)^2)^{3/2}} d\tau\right) w_{yy} + b_1 w_x + b_2 w_y, \qquad (3.5)
\end{aligned}$$

where $b_1, b_2 \in C^\infty(\overline{\Omega})$ and whose expressions are not needed to be explicit. Consider the functions defined on $\overline{\Omega}$ given by

$$a_{11} = \int_0^1 \frac{1+q(\tau)^2}{(1+p(\tau)^2+q(\tau)^2)^{3/2}} d\tau,$$
$$a_{12} = -\int_0^1 \frac{p(\tau)q(\tau)}{(1+p(\tau)^2+q(\tau)^2)^{3/2}} d\tau,$$
$$a_{22} = \int_0^1 \frac{1+p(\tau)^2}{(1+p(\tau)^2+q(\tau)^2)^{3/2}} d\tau.$$

With this choice of functions a_{ij} and b_i, define L as in (3.4). Observe that the linear operator L depends on the initial functions u_1 and u_2. By (3.5), the function w satisfies $Lw \ge 0$. We prove that the operator L is uniformly elliptic. The quadratic form

$$Q(\tau, \xi) = \big(1+q(\tau)^2\big)\xi_1^2 - 2p(\tau)q(\tau)\xi_1\xi_2 + \big(1+p(\tau)^2\big)\xi_2^2,$$

with $\xi = (\xi_1, \xi_2)$, is positive definite because

$$Q(\tau, \xi) = \xi_1^2 + \xi_2^2 + \big(q(\tau)\xi_1 - p(\tau)\xi_2\big)^2 > 0$$

for $\xi \neq (0,0)$. This proves that

$$R(\tau, \xi) = \frac{1}{(1+p(\tau)^2+q(\tau)^2)^{3/2}} Q(\tau, \xi)$$

is positive definite. Finally, we write

$$\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j = \int_0^1 R(\tau,\xi)d\tau > 0,$$

proving that L is an elliptic operator. On the other hand, the least eigenvalue of Q is 1. As the functions $p(\tau)$ and $q(\tau)$ are continuous on $\overline{\Omega}$, there exists an $M > 0$ such that $(1 + p(\tau)^2 + q(\tau)^2)^{3/2} \leq M$ on $\overline{\Omega}$. Hence, the least eigenvalue of L satisfies

$$\lambda = \lambda(x, y) = \int_0^1 \frac{1}{(1 + p(\tau)^2 + q(\tau)^2)^{3/2}}\, d\tau \geq \frac{1}{M} > 0.$$

This proves that L is uniformly elliptic. □

As an immediate consequence of the Hopf maximum principle and Lemma 3.2.3, we have:

Theorem 3.2.4 (Tangency principle) *Let M_1 and M_2 be two surfaces of $\mathbb{R}^3$ and either $p \in int(M_1) \cap int(M_2)$ or $p \in \partial M_1 \cap \partial M_2$. Assume*:

1. *$M_1 \geq M_2$ around p.*
2. *$H_1 \leq H_2$ around p.*

Then M_1 and M_2 agree on a neighborhood of p. In particular, $H_1 = H_2$ in this neighborhood.

Proof In an open set around p, M_1 and M_2 are graphs of functions u_1 and u_2 respectively, defined on $\overline{\Omega} \subset \mathbb{R}^2$. Let $p = (q, u_i(q))$. Moreover, we suppose that $N_i(p) = e_3$ and $u_1 \geq u_2$. If $q \in \partial M_i$, the hypothesis $u_1 \geq u_2$ implies

$$\frac{\partial u_1}{\partial \mathbf{n}}(q) \geq \frac{\partial u_2}{\partial \mathbf{n}}(q), \tag{3.6}$$

where $\mathbf{n}$ is the outward unit vector along $\partial\Omega$.

If $w = u_2 - u_1$, then $w \leq 0$ with $w(q) = 0$ and the function w attains a maximum at q. As $H_1 \leq H_2$, Lemma 3.2.3 says that $Lw \geq 0$. If q is an interior point, the interior version of the Hopf maximum principle implies that w is constant in $\overline{\Omega}$ and as $w(q) = 0$, we have $u_1 = u_2$ in $\overline{\Omega}$. If $q \in \partial\Omega$, inequality (3.6) implies

$$\frac{\partial w}{\partial \mathbf{n}}(q) \leq 0.$$

Taking into account the boundary version of the Hopf maximum principle, we conclude again that $u_1 = u_2$ on $\overline{\Omega}$. □

When both surfaces have the same (constant) mean curvature, we obtain:

Corollary 3.2.5 *Let M_1 and M_2 be two surfaces of $\mathbb{R}^3$ and either $p \in int(M_1) \cap int(M_2)$ or $p \in \partial M_1 \cap \partial M_2$. Assume one surface lies above the other around p and they have the same constant mean curvature. Then M_1 and M_2 agree on a neighborhood of p.*

In the literature, the tangency principle is sometimes called the 'maximum principle'. We shall use the terminology 'maximum principle' in the context of differential equations and 'tangency principle' when working with surfaces of constant mean curvature.

It is also possible to prove the tangency principle without the use of the Hopf maximum principle but we need to assume that the solutions of (3.2) are analytic [Hop83]. We only consider the interior point version and that both surfaces have the same (constant) mean curvature, which it is the interesting case. Let u_1 and u_2 be two functions that locally represent two H-surfaces tangential at a point p and where the orientations coincide at p. Without loss of generality, assume that u_1 and u_2 are defined on a domain $\Omega \subset \mathbb{R}^2$ about the origin $(0,0)$ and the tangent plane at this point is $\mathbb{R}^2$. Then

$$u_1(0,0) = u_2(0,0), \qquad \nabla u_i(0,0) = (0,0).$$

Assume $u_1 \neq u_2$ around the origin. If u_1 and u_2 satisfy (3.2), the proof of Lemma 3.2.3 implies that $w = u_2 - u_1$ satisfies a linear elliptic operator $Lw = 0$. Moreover, $Lw(0,0) = (a_{11}w_{xx+} + 2a_{12}w_{xy} + a_{22}w_{yy})(0,0)$. After an affine change of coordinates (x, y), we suppose that $Lw(0,0) = (w_{xx} + w_{yy})(0,0) = 0$. Using the analyticity of u_i, if $u_1 \neq u_2$, there is an expansion of w around $(0,0)$ such that if $z = x + iy$ and $r = |z|$, then

$$u_2 - u_1 = P(z) + O\left(r^{n+1}\right), \tag{3.7}$$

where $P(z)$ is a homogeneous harmonic polynomial of degree n. Thus $P(z)$ is the imaginary part of a holomorphic function, more precisely, $P(z) = \mathrm{Im}(cz^n)$, $c \in \mathbb{C} - \{0\}$. Furthermore, after a rotation of coordinates if necessary, $c \in \mathbb{R}$. Then the difference function $w = u_2 - u_1$ around the origin behaves as $cr^n \sin(n\theta)$, $n \geq 2$, with $z = re^{i\theta}$. In particular, we have proved that in a neighborhood U of $(0,0)$, the set $w^{-1}(\{0\})$ has at least four smooth arcs intersecting at $(0,0)$ and $U \setminus w^{-1}(\{0\})$ is formed at least by four components on which the sign of w alternates. We conclude that, if $u_1 \geq u_2$ around $(0,0)$, then necessarily $u_1 = u_2$ in a neighborhood of the origin.

A direct consequence of the tangency principle is the uniqueness of graphs with the same mean curvature and the same boundary.

Theorem 3.2.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. If M_1 and M_2 are two H-graphs on Ω with $\partial M_1 = \partial M_2$, then $M_1 = M_2$.*

In the language of differential equations, this result is the uniqueness of solutions of the Dirichlet problem (1.5)–(1.6). Indeed, if $M_i = \mathrm{graph}(u_i)$, $i = 1, 2$, are two

solutions, then $L(u_1 - u_2) = 0$. By the Hopf maximum principle, $u_1 - u_2$ attains its maximum and its minimum at $\partial\Omega$. But $u_1 = u_2$ along $\partial\Omega$, proving $u_1 = u_2$ in Ω.

We shall prove the result by means of the tangency principle, giving a geometric proof of the result. In fact, since the tangency principle is a consequence of the maximum principle, we shall actually express in a geometric language the uniqueness of solutions of Eq. (3.2). The existence of solutions of the Dirichlet problem will be considered in Chap. 8.

Proof We point out that in the statement of the result, both graphs have the same orientation, say for example, given by (3.1), because they satisfy the same Eq. (3.2). Consider the vertical translations $\phi_t(p) = p + te_3$. Move M_1 up by ϕ_t until a time $t_0 > 0$ such that $\phi_{t_0}(M_1) \cap M_2 = \emptyset$. Next decrease t until $\phi_t(M_1)$ first makes contact with M_2 at $t = t_1$. Necessarily, this occurs for some $t_1 \geq 0$. If $t_1 > 0$, then $\phi_{t_1}(M_1)$ and M_2 intersect at an interior point p. Since both surfaces have the same orientation and mean curvature at p, both surfaces agree on an open set around p by the tangency principle. We repeat the argument, proving that the set where $int(\phi_{t_1}(M_1))$ and $int(M_2)$ coincide is open both in $int(\phi_{t_1}(M_1))$ and in $int(M_2)$. Since this set is always closed, $int(\phi_{t_1}(M_1)) = int(M_2)$ and so, $\phi_{t_1}(M_1) = M_2$. As $\partial\phi_{t_1}(M_1) \neq \partial M_2$, we get a contradiction.

In conclusion, $t_1 = 0$ and this shows that M_1 lies above M_2. Reversing the roles of M_1 and M_2, the same argument shows that M_2 lies above M_1, and we conclude that $M_1 = M_2$. □

Remark 3.2.7 If M is an H-graph on Ω of a function u, with $u = 0$ on $\partial\Omega$, its reflection M^* with respect to $\mathbb{R}^2$ is the graph of $-u$ and with the same boundary. If the mean curvature H is computed with respect to upward orientation, then u satisfies (3.2). However, the function $-u$ satisfies (3.2) for the value $-H$. This is consistent with the previous result. Moreover, if $H = 0$, this argument and the uniqueness prove $u = -u$. Thus $u = 0$, rediscovering Theorem 3.1.12.

3.3 Applications of the Tangency Principle

In this section we illustrate examples of the use of the tangency principle in the study of cmc surfaces with boundary.

Theorem 3.3.1 *A compact minimal surface immersed in $\mathbb{R}^3$ is included in the convex hull of its boundary.*

Proof In view of Corollary 3.1.13, we know that $\partial M \neq \emptyset$. If the boundary is a plane curve, we complete the proof using Theorem 3.1.12. Assume now that the boundary is not planar. Let $P = \langle a \rangle^{\perp}$ be a plane disjoint with M, where $\langle a \rangle$ is the line generated by the vector a. Move P in the direction of the vector a and towards $x(M)$ until the first touching point p. Let P' denote the plane P at this position. If

$p \in int(M)$, the tangency principle implies that $x(M)$ agrees with P' in an open set around p. By Theorem 2.2.2, $x(M) \subset P'$, which is not possible because ∂M is not a plane curve. It follows that $p \in \partial M$. Thus, given a plane P, there exists a parallel plane P' to P such that $x(M) \cap P' \subset x(\partial M)$ and $x(M)$ lies on one side of P'. This gives the theorem. □

This result provides a proof of the following empirically evident fact concerning soap films. When we construct a soap film with a skeleton of wire and soapy solution, we cannot touch the soap film with the palm of our hand but we can touch the wire. In this case, when we attempt to touch the soap film, the palm of our hand represents a plane, a minimal surface, which, by the tangency principle, cannot contact the soap film at interior points.

We study how the value H in the constant mean curvature equation (3.2) imposes restrictions on the shape and the size of the domain Ω. In other words, given a domain $\Omega \subset \mathbb{R}^2$ and a number H, the existence of an H-graph on Ω depends on a certain relation between Ω and H. This can be seen in the next two results for cmc graphs. The first one is due to S.N. Bernstein [Brn15] (see also [Fin65]).

Theorem 3.3.2 *Let M be an H-graph on a domain Ω with $H \neq 0$. Then Ω does not contain the closure of a disk of radius $1/|H|$.*

Proof We proceed by contradiction. Assume that D is an open disk of radius $1/|H|$ such that $\overline{D} \subset \Omega$. Let p be the center of D. Reversing the orientation on M if necessary, suppose $H > 0$. Assume that the orientation points upwards. Let S be a sphere of radius $1/H$ whose center lies on the straight-line through p perpendicular to the (x, y)-plane and we orient S with the inward orientation. With this choice of orientation, the mean curvature of S is H and the orthogonal projection of S on $\mathbb{R}^2$ is $\overline{D}$.

Lift S vertically upwards until S is completely above M. Then, let us descend S until the first point of contact between S and M. Since $\overline{D} \subset \Omega$ and M is a graph on Ω, the contact point p must be interior in M. By the tangency principle, the surfaces M and S agree on an open set around p. Theorem 2.2.2 implies that M is included in a sphere of radius $1/H$, say, $M \subset S$. This is a contradiction because the orthogonal projection onto $\mathbb{R}^2$ would give $\overline{\Omega} \subset \overline{D}$.

In the case that the orientation of M points downwards (for $H > 0$), we follow a similar argument with the sphere S approaching from below. □

Corollary 3.3.3 *There do not exist graphs defined on $\mathbb{R}^2$ with non-zero constant mean curvature.*

In contrast to the statement of this corollary, and in the case $H = 0$, the plane is the only minimal graph defined on $\mathbb{R}^2$. This is the famous Bernstein theorem, but its proof has a different flavor than Theorem 3.3.2 [Alm66b, Brn15, Hop52, Oss86]. On the other hand, there exist non-planar minimal graphs defined on arbitrary large

domains, as for example, the Scherk surface. This surface is the graph of

$$z = u(x, y) = \frac{1}{r} \log \left| \frac{\cos(ry)}{\cos(rx)} \right|, \quad |x|, |y| < \frac{\pi}{2r},$$

defined on a square of arbitrary size letting $r \to 0$. In fact, we take any minimal graph M defined on a bounded domain $\Omega \subset \mathbb{R}^2$. Applying homotheties of M, i.e., tM, the surface is minimal again, but now tM is a graph on the domain $t\Omega$, which can be arbitrarily large as we increase the value of t.

The second result is a simple application of the divergence theorem. It tells us that given a bounded domain $\Omega \subset \mathbb{R}^2$, the existence of solutions of the Dirichlet problem associated to Eq. (3.2) does not admit arbitrary values of H, but there are a priori restrictions on H depending on the size of Ω and $\partial\Omega$.

Theorem 3.3.4 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $H \in \mathbb{R}$. If M is an H-graph of $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, then*

$$|H| \leq \frac{L(\partial\Omega)}{2a(\Omega)},$$

where $L(\partial\Omega)$ is the length of $\partial\Omega$ and $a(\Omega)$ is the area of Ω.

Proof Let us first assume that $u \in C^2(\overline{\Omega})$. If $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, an integration of (3.3) gives

$$2|H|a(\Omega) = \left| \int_{\partial\Omega} \left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \mathbf{n} \right\rangle ds \right| < \int_{\partial\Omega} 1 \, ds = L(\partial\Omega). \tag{3.8}$$

If we now suppose that u is only continuous along $\partial\Omega$, take an increasing exhaustion of domains $\Omega_n \to \Omega$, with $\overline{\Omega_n} \subset \Omega$. If $u_n = u_{|\Omega_n}$, then $u_n \in C^2(\overline{\Omega_n})$ and (3.8) gives $2|H|a(\Omega_n) < L(\partial\Omega_n)$. Letting $n \to \infty$, we obtain the result. □

If Ω is a round disk of radius r, this result asserts that $|H| \leq 1/r$. This is consistent with the family of small spherical caps and the hemisphere with boundary $\partial\Omega$.

Another application of the tangency principle provides an estimate of the size of an H-surface spanning a given curve Γ depending on the size of Γ. Denote by $B(q, r) \subset \mathbb{R}^3$ an open ball of radius r centered at q and let $S(q, r) = \partial B(q, r)$ be the boundary sphere. The next theorem is due to J. Serrin [Ser69a].

Theorem 3.3.5 *Let Γ be a closed curve contained in $\overline{B(q, 1)}$ and $H \neq 0$. If M is a compact H-surface immersed in $\mathbb{R}^3$ spanning Γ such that $M \subset \overline{B(q, 1/|H|)}$, then $M \subset \overline{B(q, 1)}$.*

Proof We orient M by the Gauss map N such that $H > 0$. If $H \geq 1$, the result is trivial. Thus, let $0 < H < 1$. Let us assume for contradiction that $M \not\subset \overline{B(q, 1)}$.

Since M is compact, let $r_0 = \min\{r > 1 : M \subset \overline{B(q,r)}\} > 1$. The mean curvature of $S(q, r_0)$ is $1/r_0$ with the inward orientation and, by hypothesis, $r_0 \leq 1/H$.

Let $p \in S(q, r_0) \cap M$. If $p \in \Gamma$, then $B(q, r_0) \subset B(q, 1)$ and so, $r_0 \leq 1$, a contradiction. Consequently, p is an interior point of M. In particular, $S(q, r_0)$ and M are tangent at p. Since $M \subset \overline{B(q, r_0)}$ and $H > 0$, the comparison principle implies that $N(p)$ points inside the ball $B(p, r_0)$ and $H \geq 1/r_0$. Thus $H = 1/r_0$ and the tangency principle implies that M and $S(q, r_0)$ agree on an open set around p. By Theorem 2.2.2, $M \subset S(q, r_0)$. This contradicts that $\Gamma \subset \overline{B(q, 1)}$ and $\overline{B(q, 1)} \cap S(q, r_0) = \emptyset$ since $r_0 > 1$. □

We devote the remainder of this section to some results that tell us about the limitation of a cmc surface inside a region defined by cylinders and spheres whose radii have an appropriate relation to the mean curvature of the surface.

Theorem 3.3.6 *Let $H \neq 0$. The only closed immersed H-surface in $\mathbb{R}^3$ included in the closure of a solid right-cylinder of radius $1/|H|$ is a round sphere.*

Proof Let M be a closed H-surface under the assumptions. After a change of coordinates, assume that M is included in the cylinder $K = D \times \mathbb{R}$, where $D \subset \mathbb{R}^2$ is a closed disk of radius $1/|H|$ and let $C = \partial D \times \mathbb{R}$. Consider an upper hemisphere S of radius $1/|H|$ such that $\partial S = \partial D$, in particular, $S \subset K$. We orient S with the downward orientation N_S and so the mean curvature of S is $|H|$. Since M and S are compact surfaces, we can move S up until it does not intersect M. Now, let us move S down until it first intersects M at a point p. The point p is a tangent point of both surfaces. Indeed, p is always an interior point of M because M is closed. If $p \in int(S)$, then p is a common interior point, in particular, it is a tangent point. If $p \in \partial S$, then T_pS is a vertical plane and $T_pS = T_pC$. As M is included in K and $p \in C = \partial K$, then M and C are again tangent at p.

Once that S touches M at p, we orient M by the Gauss map N such that $N(p) = N_S(p)$. Since p is the first contact point, we see that $M \geq S$ around p. This occurs even if $p \in C$ because S and M are graphs in a domain of T_pC. The comparison principle says that the mean curvature of M is greater than that of S, and as the mean curvature of S is $|H|$, we deduce that the mean curvature of M is positive and so it agrees with $|H|$.

Since the mean curvatures of M and S coincide, the tangency principle implies that M and S coincide on an open set around p. Taking into account Theorem 2.2.2 and since M is a closed surface, we conclude that M is a sphere of radius $1/|H|$ and this proves the result. □

Corollary 3.3.7 *Let $H \neq 0$. The only closed immersed H-surface in $\mathbb{R}^3$ included in the closure of a round ball of radius $1/|H|$ is the round sphere.*

The comparison of the size of a surface with spheres in terms of the mean curvature can be generalized when H is not constant.

Corollary 3.3.8 *Let M be a closed surface immersed in $\mathbb{R}^3$ whose mean curvature function H satisfies $|H| \leq h$, $h > 0$. If M is included in the closure of a solid right-cylinder of radius r, then $r \geq 1/h$ and equality holds if and only if M is a sphere of radius $1/h$.*

The particular case of this result assuming that the surface is included in a sphere was proven by D. Koutroufiotis in [Kou73].

Theorem 3.3.6 and Corollary 3.3.7 refer to immersed H-surfaces where two or three spatial coordinates are bounded in terms of H. Assume now that one coordinate of the surface is bounded in relation with H. We need the next definition. A *slab* is the region of $\mathbb{R}^3$ lying between two parallel planes and its width is the distance between both planes. Fix $H \neq 0$. Since a sphere of radius $1/|H|$ is included in a slab of width $2/|H|$, the possible extension of the previous results for slabs should assume that the width is less than or equal to $2/|H|$, in contrast to the value $1/|H|$ that appears in the statements of Theorem 3.3.6 and Corollary 3.3.7. However, we have:

Theorem 3.3.9 *Let $H \neq 0$. There do not exist closed immersed H-surfaces in $\mathbb{R}^3$ included in the closure of a slab of width $1/|H|$.*

Proof Without loss of generality, we suppose $H > 0$. Assume that there exists a closed H-surface M included in a slab of width $1/H$. Suppose that the slab is horizontal, which means that the planes P_1 and P_2 that define the slab are parallel to the plane $z = 0$. To a horizontal vector v, we associate a half cylinder C_v with the following properties (by a half cylinder we mean the closure of either of the two connected components of an infinite cylinder of circular cross-section that separates a plane through its axis):

1. C_v is included in the slab and the radius of C_v is $1/(2H)$.
2. The axis of C_v is perpendicular to v.
3. Each of the connected components of ∂C_v, say, two parallel straight lines, is included in each of the planes P_1 and P_2.

The surface C_v separates the slab S into two connected components. See Fig. 3.4. Let C_v^+ denote the convex component and orient C_v by N' pointing towards C_v^+. It follows that the mean curvature of C_v is positive, say, $H > 0$. In fact, C_v^+ is the mean convex side of C_v, i.e., the domain towards which the mean curvature vector of C_v points, or in other words, where the orientation corresponding to the positive mean curvature points.

Move C_v by translations in the v-direction in such way that M lies in C_v^+. Displace C_v in the opposite direction until it touches M. If p is a first touching point, then p is a tangent point of M and C_v, even if $p \in \partial C_v$. Now we orient M by the Gauss map so that $N(p) = N'(p)$. With this choice of N, we have $M \geq C_v$ around p and the comparison principle asserts that the mean curvature of M is greater than or equal to H. This proves that the mean curvature of M is, indeed, H according to this orientation. The tangency principle now concludes that M and C_v agree around p.

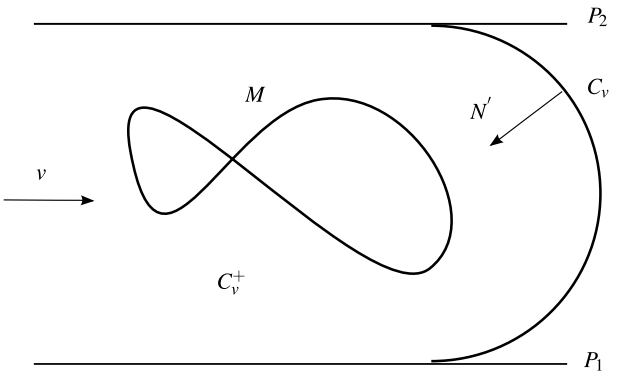

Fig. 3.4 Proof of Theorem 3.3.9. The comparison and the tangency principle apply between the surface M and the half cylinder C_v

Extending the argument, $M \cap C_v$ is an open and closed set of M, hence $M \subset C_v$. This is a contradiction because M is a closed surface. □

In view of Theorem 3.3.9, it remains an open question to determine whether the assumption that the width of the slab can be relaxed to be $2/|H|$, as is the case for a sphere with constant mean curvature H.

Chapter 4
Constant Mean Curvature Embedded Surfaces

In this chapter we are interested in the question of what type of symmetries are imposed on the shape of a compact cmc surface M if it has as a prescribed boundary some closed curve $\Gamma \subset \mathbb{R}^3$. To be precise, let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry such that Γ is invariant under Φ, that is, $\Phi(\Gamma) = \Gamma$. If M is a compact cmc surface spanning Γ, we ask whether $\Phi(M) = M$. It is difficult to answer this question in all its generality. For example, if we take Γ to be circle, we expect that a cmc surface spanning Γ to be a surface of revolution. However, recall the examples discovered by N. Kapouleas of cmc surfaces bounded by a round circle that are not rotational surfaces [Kap91].

In the context of embedded surfaces, this question may be partially answered using the tangency principle. This was employed by A.D. Alexandrov in his proof that the only closed embedded cmc surface is the round sphere [Ale56, Ale62]. The technique has been used profusely in the theory of embedded cmc surfaces, as predicted by H. Hopf in [Hop83, p. 147]:

> "It is my opinion that his proof by A.D. Alexandrov (...) opens important new aspects in differential geometry in the large."

In this chapter we focus on compact embedded cmc surfaces. A direct consequence of the Alexandrov's method is a partial answer to Conjecture 2: we shall prove that a compact embedded cmc surface with circular boundary and included in one side of the boundary plane must be a spherical cap. Then the conjecture reduces to the question of what conditions guarantee that such a surface lies on one side of the boundary plane.

Besides results of symmetry, the Alexandrov's technique allows us to obtain information on the shape of an embedded cmc surface depending on the geometric properties of its boundary. When the boundary is planar, and by using the Alexandrov method with planes parallel to the plane containing the boundary, we shall prove in some cases that the surface is a graph. Finally, we deal with the case where the boundary curve is included in a sphere and similarly, we shall ask when the surface lies on one side of this sphere.

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_4,
© Springer-Verlag Berlin Heidelberg 2013

4.1 The Alexandrov Reflection Method

The use of planes to reflect a cmc surface and taking the same surface for comparison with itself is the so-called *Alexandrov reflection method*, also called in the literature the *method of moving planes*. In order to illustrate this method, we prove the boundary version of the Alexandrov theorem [Koi86, Sch83].

Theorem 4.1.1 *Let Γ be a simple closed curve contained in a plane P. Assume that there exists a plane Π orthogonal to P such that Γ is symmetric with respect to Π. Let $L = P \cap \Pi$ and assume that each of the two connected components of $\Gamma \setminus (\Gamma \cap L)$ is a graph on L. Let M be a compact embedded cmc surface in $\mathbb{R}^3$ with boundary Γ. If $int(M)$ lies on one side of P, then*:

1. *The surface M is symmetric with respect to Π.*
2. *Each connected component of $M \setminus (M \cap \Pi)$ is a graph on Π.*

Proof Denote by $\Omega \subset P$ the domain bounded by Γ. Since the surface M is embedded and lies on one side of P, $M \cup \Omega$ bounds a 3-domain $W \subset \mathbb{R}^3$ and the mean curvature is not zero. Choose on M the orientation N pointing towards W and let H be the mean curvature of M computed with respect to N. After a rigid motion, assume that P and Π are the planes $z = 0$ and $y = 0$, respectively. Consider the 1-parameter family of planes Π_t, $t \in \mathbb{R}$, of equation $y = t$. We introduce the following notation. If $A \subset \mathbb{R}^3$, let $A_t^+ = A \cap \{(x, y, z) \in \mathbb{R}^3 : y \geq t\}$, $A_t^- = A \cap \{(x, y, z) \in \mathbb{R}^3 : y \leq t\}$ and denote by A_t^* the reflection of A_t^+ with respect to Π_t.

Consider the planes Π_t coming from $t = +\infty$, which do not intersect M by the compactness of M. Approaching from the right ($t \searrow 0$), we arrive at the first plane Π_{t_0}, $t_0 > 0$, that touches M at a point $q \in M$.

Claim 1. There exists an $\varepsilon > 0$ such that for $t \in (t_0 - \varepsilon, t_0)$:

1. $M_t^* \setminus \partial M_t^* \subset W$; and
2. M_t^+ is a graph on Π_t.

We distinguish two cases. If $q \in int(M)$, then $T_q M = \Pi_{t_0}$, M is a graph on Π_{t_0} around q and Π_t intersects M transversally for $t \in (t_0 - \varepsilon, t_0)$ for some $\varepsilon > 0$. Since M is embedded, $M_t^* \setminus \partial M_t^* \subset W$. If $q \in \Gamma$, then $T_q \Gamma$ is parallel to L and Γ is a graph on L around q. This means that the intersection of Π_t with Γ around Γ is a graph on L for $t \in (t_0 - \varepsilon, t_0)$. This proves the claim. In addition, in the interval $(t_0 - \varepsilon, t_0)$, the mean curvature of the reflected surface M_t^* is H with respect to the orientation N^* induced by the symmetry about Π_t. Here for each $t \in (t_0 - \varepsilon, t_0)$, $\partial M_t^+ = \partial M_t^*$ and, depending on whether Π_t intersects or does not intersect Ω, $M_t^+ \cup M_t^*$, together with a subdomain of Ω if necessary, defines a 3-domain $W_t \subset \mathbb{R}^3$ and is included in W. In particular, N^* points towards W_t.

Decrease t and let

$$t_1 = \inf\{t \in \mathbb{R} : M_s^* \setminus \partial M_s^* \subset W, M_s^* \text{ is a graph on } \Pi_s, \forall s \in (t, t_0)\}. \tag{4.1}$$

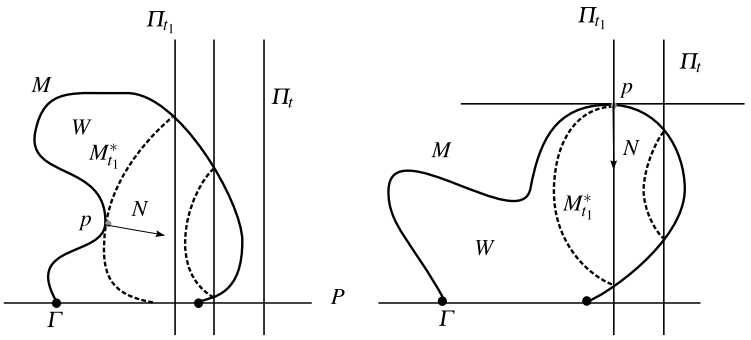

Fig. 4.1 The Alexandrov reflection method in Theorem 4.1.1

By the compactness of M, this infimum exists and because Π is a plane of symmetry of Γ, $t_1 \geq 0$. The number t_1 satisfies $M_t^* \not\subset \overline{W}$ for $t < t_1$. We say that t_1 is the first time t such that $M_t^* \not\subset \overline{W}$ and this expression will be used in the remainder of the book.

There are two possibilities depending on the value of t_1:

(i) $t_1 > 0$. In this case one of the properties $M^*_{t_1} \setminus \partial M^*_{t_1} \subset W$ or $M^*_{t_1}$ is a graph on Π_{t_1} fails. In the first case, there exists a $p \in int(M^*_{t_1}) \cap int(M^-_{t_1})$ (Fig. 4.1, left). In the second case, there exists a $p \in \partial M^*_{t_1} \cap \partial M^-_{t_1}$ where $M^*_{t_1}$ and $M^-_{t_1}$ are tangent at p (Fig. 4.1, right).

(ii) $t_1 = 0$.

Claim 2. Case (i) is not possible.

Since each connected component of $\Gamma \setminus (\Gamma \cap L)$ is a graph on L and $t_1 > 0$, we must have $p \notin \Gamma$. In the first possibility, namely, $p \in int(M^*_{t_1}) \cap int(M^-_{t_1})$, the surfaces $M^*_{t_1}$ and $M^-_{t_1}$ are tangent at p because the point p is interior and $t = t_1$ is the first time that $int(M^*_t)$ contacts with ∂W, $t \geq t_1$. On the other hand, $M^*_{t_1} \geq M^-_{t_1}$ around p because $N(p)$ and $N^*(p)$ agree, since both are pointing towards W. In the second possibility, the point p belongs to P_{t_1}. Since $M_t^* \not\subset \overline{W}$ for $t < t_1$, then $M^*_{t_1}$ and $M^-_{t_1}$ are tangent at p and T_pM is orthogonal to P_{t_1}. Moreover, both surfaces $M^*_{t_1}$ and $M^-_{t_1}$ are graphs in a domain in the closed half plane $(T_pM)^-$. Again, the vectors $N(p)$ and $N^*(p)$ agree and $M^*_{t_1} \geq M^-_{t_1}$, where now $N(p) \in \Pi_{t_1}$.

We apply the tangency principle in its interior point version in the first case and the boundary point version in the second case. In both settings, $M^*_{t_1}$ and $M^-_{t_1}$ agree on an open set around p. Now we make use of an argument that appeared in Theorem 3.3.2, proving that $M^*_{t_1}$ and $M^-_{t_1}$ agree on a maximal domain. In fact, $M^*_{t_1} \cap M^-_{t_1}$ is a sub-surface included both in $M^*_{t_1}$ and $M^-_{t_1}$. Let K' denote the connected component of $M^*_{t_1}$ containing p and let K be the corresponding component of $M^+_{t_1}$ such that $K' = K^*_{t_1}$. Since $M^-_{t_1}$ is a closed set in M, and extending the Alexandrov argument, $K' \cap M^-_{t_1}$ is an open and closed set of K', hence $K' \subset M^-_{t_1}$. Thus, $K \cup K'$ is a compact surface whose boundary is a simple closed curve included in Γ. By

connectedness, $K \cup K' = M$. This proves that $M^*_{t_1} = M^-_{t_1}$ and M is invariant under the reflection about Π_{t_1}. In particular, Π_{t_1} is a plane of symmetry of Γ, which is a contradiction because $t_1 > 0$.

As a consequence of the two claims, the reflection method about the planes Π_t continues until the position $t_1 = 0$ and so the plane Π_{t_1} is the very plane Π. In this setting there are two possibilities:

1. The plane Π is a plane of symmetry of M. Then the result is proved, because by the definition of t_1, the surface M^+_t is a graph on Π_t for all $t > 0$.
2. The plane Π is not a plane of symmetry of M. Then by the above argument, $M^*_0 \setminus \partial M^*_0 \subset W$, M^*_0 and M^-_0 are not tangent at points of Γ and for any $p \in \partial M^*_0 \setminus \Gamma$, the tangent plane T_pM is not perpendicular to Π. We discard this case. We begin with the Alexandrov process from the left, i.e., for large negative values of t. If $A \subset \mathbb{R}^3$, we use the notation A^t_* to indicate the reflection of A^-_t with respect to Π_t. By means of the planes Π_t, we approach M until Π_t first touches M at $t = t_2 < 0$. Next, we continue to the right and reflect M^-_t with respect to Π_t. For values $t > t_2$ close to t_2, we have a result analogous to claim 1 above. Next displace Π_t by letting $t \to 0$ and reflecting M^-_t about Π_t until the first time $t = t_3$ such that $M^t_* \not\subset \overline{W}$. Let

$$t_3 = \sup\left\{t \in \mathbb{R} : M^s_* \setminus \partial M^s_* \subset W, \text{for all } s \in (t_2, t)\right\}.$$

By the earlier assumption, $t_3 < 0$. Then an argument as in claim 2 gives that Π_{t_3} is a plane of symmetry, a contradiction.

□

We make some observations:

1. In the Alexandrov method, let t_1 be the first time that $M^*_{t_1} \not\subset \overline{W}$. Assume $p \in int(M^*_{t_1}) \cap int(M^-_{t_1})$ is a tangent point and denote by $q \in M^+_{t_1}$ the point such that $q^* = p$. Then $p \neq q$ and by the process, the horizontal open segment joining p and q, say (p, q), must be included in W.
2. We may assume in Theorem 4.1.1 that Γ is disconnected. Indeed, let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$ be the decomposition into connected components and let $\Omega_i \subset P$ denote the domain bounded by Γ_i. Since each piece of $\Gamma \setminus (\Gamma \cap L)$ is a graph on L, then $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$. If $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$, then $M \cup \Omega$ bounds a 3-domain W and the proof now follows the same steps.
3. If the surface has points in both sides of the boundary plane P, the proof of Theorem 4.1.1 fails because it could happen that the contact point p in claim 2 is not a tangent common point, as for example, if $p \in \Gamma$: see Fig. 4.2, left.
4. Another setting where the Alexandrov process fails occurs if the curve Γ is not a graph on L. In such case, it could happen that $p \in \Gamma$ and p is not a tangent point of the surfaces $M^*_{t_1}$ and $M^-_{t_1}$ as depicted in Fig. 4.2, right. In the case of the figure, the boundary curve is not symmetric with respect to L. Even if Γ is symmetric with respect to Π, there could exist an embedded cmc surface M with boundary Γ, where M included in one side of P but M is not symmetric with

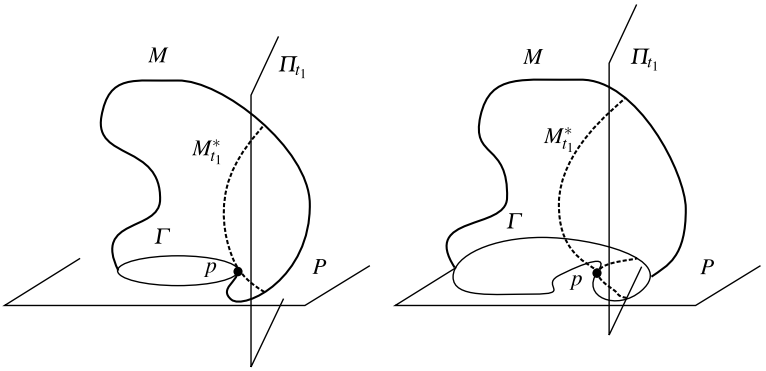

Fig. 4.2 Two cases where the Alexandrov reflection method fails. *On the left*, p is a boundary point which is the reflection of an interior point of $M^+_{t_1}$. *On the right*, p is the reflection of a boundary point of $M^+_{t_1}$. In both cases, $M^-_{t_1}$ and $M^*_{t_1}$ are not tangent at p

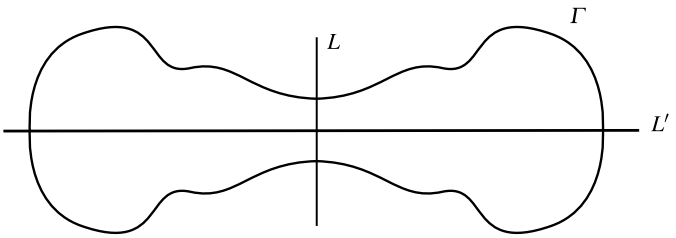

Fig. 4.3 A simple closed curve Γ contained in a plane P that is symmetric about L and L'. A compact embedded cmc surface spanning Γ and included in one side of P could not be symmetric about the reflection with respect to the plane containing L and orthogonal to P. However, by Theorem 4.1.1, it must be symmetric about the plane orthogonal to P and containing L'

respect to Π. This would be the case if each of the pieces of Γ that separates L is not a graph on L: for an example of a possible boundary curve see Fig. 4.3 [Sch83].

Given a strictly convex plane curve Γ, if Γ is symmetric about a straight line L, then L separates Γ into two convex sub-arcs that are graphs on L. Then:

Corollary 4.1.2 *Let M be a compact embedded cmc surface in $\mathbb{R}^3$ spanning a strictly convex plane curve. If M lies on one side of the plane containing Γ, then M inherits the symmetries of Γ.*

In the particular case that the boundary is a circle, we obtain:

Corollary 4.1.3 *Let M be a compact embedded cmc surface in $\mathbb{R}^3$ spanning a circle Γ. If M lies on one side of the plane containing Γ, then M is a spherical cap.*

Proof Let R denote the straight line orthogonal to P containing the center of Γ. By Corollary 4.1.2, the surface is symmetric with respect to any plane containing R. This shows that M is rotationally symmetric about R. Then M is a planar disk or a spherical cap. Since the surface lies on one side of P, M is a spherical cap. □

If the surface is not compact, we can propose similar questions of symmetry, as for example, assuming that M is an embedded cmc surface with boundary a circle Γ. Examples of such surfaces appear when we cut a catenoid or an unduloid by a plane perpendicular to the axis of rotation and take one of the two components. However the application of the reflection technique may present difficulties. This occurs for the half catenoid $z = \operatorname{arc}\cosh(\sqrt{x^2+y^2})$, $x^2+y^2 \geq 1$. This is a minimal surface with boundary the circle $\mathbb{S}^1$ in the plane $z = 0$. Here we cannot even start the reflection process because any vertical plane Π_t intersects the surface. For an unduloid, the surface is cylindrically bounded but in the process of reflection, it may occur that the tangent point of contact lies 'at infinity'. In this setting, R. Earp and H. Rosenberg consider a properly embedded, complete surface in $\mathbb{R}^3$ with non-zero constant mean curvature and with boundary a strictly convex curve Γ included in a plane P. They prove that if the surface lies on one side of P and has only one end orthogonal to P, that is, outside of a compact set, and the surface is contained in a cylinder orthogonal to P, then the surface inherits the symmetries of its boundary [ER91]. In particular, if the boundary is a circle, the surface is a piece of a Delaunay surface. The proof uses the Alexandrov method by tilted planes but close enough to be orthogonal to P.

Returning to the compact case, if the boundary is a circle, conjecture 2 reduces to the case where the surface lies on one side of the boundary plane. As a consequence of Theorem 3.1.11 we have [Koi86]:

Corollary 4.1.4 *Let Γ be a circle included in a plane P and denote by $\Omega \subset P$ the open disk bounded by Γ. If M is a compact embedded cmc surface in $\mathbb{R}^3$ spanning Γ and $M \cap (P \setminus \overline{\Omega}) = \emptyset$, then M is a spherical cap.*

If the surface is closed, we may use the proof of Theorem 4.1.1 to prove the original Alexandrov theorem.

Corollary 4.1.5 (Alexandrov) *The only closed embedded cmc surface in Euclidean space is the round sphere.*

Proof We proceed as in the proof of Theorem 4.1.1 taking into account that there are no boundary points. The proof shows that given a direction $v \in \mathbb{R}^3$, there exists a plane Π_v orthogonal to v such that M is invariant by reflection through Π_v. The only closed embedded surface in $\mathbb{R}^3$ with this property is the round sphere (a proof of this fact can be found in [Hop83, p. 147] and [Spi79, p. 509]). □

The Alexandrov theorem is still valid in space forms by using reflections about totally geodesic surfaces. For hyperbolic space $\mathbb{H}^3$, see Corollary 10.3.3. In the

3-dimensional sphere $\mathbb{S}^3$, it is necessary that the surface is included in a hemisphere of $\mathbb{S}^3$. This ensures that there exists a totally geodesic surface, say, $\mathbb{S}^2 \times \{0\}$, that does not intersect the surface and thus, we can start with the reflection process.

For closed surfaces, we can consider other assumptions on the mean curvature. In this setting, H.C. Wente proved [Wen82]:

Theorem 4.1.6 *Let $M \subset \mathbb{R}^3$ be a closed embedded surface and assume that the mean curvature function depends only on the (x, y)-coordinates, $H(x, y, z) = H(x, y)$ for all $(x, y, z) \in M$. Then there exists a horizontal plane P such that M is invariant by reflection about P. Moreover, each part of M divided by P is a graph on P.*

An example of such an assumption on H is

$$H(x, y, z) = a\left(x^2 + y^2\right) + b, \quad a, b \in \mathbb{R}.$$

A surface of $\mathbb{R}^3$ whose mean curvature satisfies this condition is a mathematical model of a drop of an incompressible fluid surrounded by a rigidly rotating incompressible fluid about a fixed straight-line L with constant angular velocity ω. The equilibrium is obtained by the balance between the capillary force that comes from the surface tension γ of the drop-fluid interface S and the centrifugal force of the rotating liquid. The Laplace equation (1.1) is written here as

$$\gamma H(p) = \rho\omega^2 r(p)^2 + \lambda \quad p \in S,$$

where $r(p)$ is the distance from p to the rotation axis L, ρ is the density difference of both liquids and the constant λ results from the volume constraint. See for example [BS80, CF80, Cha65, Lan02, Lop10b, Poi85, Rss68].

The Alexandrov method may be used without assuming symmetries on the boundary curve, controlling the shape of an embedded cmc surface.

Proposition 4.1.7 *Let M be a compact embedded H-surface, $H \neq 0$. Suppose that $\partial M = \Gamma$ is a simple closed curve included in a plane $P = \{p \in \mathbb{R}^3 : \langle p, a\rangle = 0\}$, $a \in \mathbb{R}^3$, and let $\Omega \subset P$ be the domain bounded by Γ. If $M \subset \{p \in \mathbb{R}^3 : \langle p, a\rangle \geq 0\}$, then $int(M) \cap \Omega = \emptyset$ and $M \cap (P \setminus \overline{\Omega^c}) = \emptyset$, where Ω^c is the convex hull of Ω. In the particular case that Γ is convex, we have $int(M) \cap P = \emptyset$.*

Proof We know by Corollary 3.1.10 that $int(M) \cap \Omega = \emptyset$. Consider the domain W bounded by the closed surface $M \cup \Omega$. Let us suppose for contradiction that there exists a $p \in M \cap (P \setminus \overline{\Omega^c})$. Let $L \subset P$ be a straight-line that separates p from $\overline{\Omega^c}$ and let $v \in P$ be a vector orthogonal to L. Consider the 1-parameter family of planes Π_t orthogonal to v, where t indicates the signed distance to L and assume that $p \in \Pi_{\bar{t}}$ for $\bar{t} > 0$. We use the Alexandrov method as in Theorem 4.1.1 approaching from $t = +\infty$. Since Ω^c is convex, the same argument shows that the value t_1 occurs for $t_1 \geq \bar{t}$. If $t_1 = \bar{t}$, $p \in \partial M^*_{t_1} \cap \partial M^-_{t_1}$ and the surface $M^*_{t_1}$ is tangent to $M^-_{t_1}$ at p because

p is interior to M with $T_pM = P$. However, if $t_1 > \bar{t}$ or $t_1 = \bar{t}$, the Alexandrov technique proves that Π_{t_1} is a plane of symmetry of M, contradicting the fact that Γ lies on one side of Π_{t_1}. □

A result comparable with Corollary 4.1.3 concerns compact cmc surfaces with positive Gauss curvature. If M is a closed cmc surface with $K > 0$, then M encloses a convex domain of $\mathbb{R}^3$, in particular, it is star-shaped and Theorem 2.3.7 implies that M must be a round sphere. If we now assume that ∂M is a circle, we have:

Corollary 4.1.8 *The only compact cmc surface with positive Gauss curvature spanning a circle is the spherical cap.*

This result is a direct consequence of Corollary 4.1.3 and the following lemma, which does not assume that the mean curvature is constant [HRS92].

Lemma 4.1.9 *Let M be a compact surface immersed in $\mathbb{R}^3$ with $K > 0$ spanning a closed curve included in a plane P. Then M is a topological embedded disk that meets P transversally and M lies on one side of P.*

Proof Without loss of generality, assume that P is the plane $z = 0$ and identify ∂M with a closed curve $\Gamma \subset P$. Let N be the Gauss map of M. As $K \neq 0$, the normal curvature $\kappa_n(s) = \sigma(\alpha'(s), \alpha'(s)) = \kappa(s)\langle N(s), \mathbf{n}(s)\rangle$ does not vanish on Γ, where $\alpha = \alpha(s)$ is an arc length parametrization of Γ, σ is the second fundamental form of M, $\mathbf{n}$ is the unit normal vector to Γ and κ is the curvature of Γ. As $\mathbf{n} \in P$, the curve Γ is strictly convex and M meets P transversally. By the Gauss-Bonnet formula and $\kappa_n \neq 0$, we obtain

$$0 < \int_M K \, dA = 2\pi\chi(M) - \int_\Gamma \kappa_g \, ds < 2\pi\chi(M) + \int_\Gamma \kappa(s) \, ds = 2\pi\big(1 + \chi(M)\big).$$

Then $\chi(M) > -1$. As

$$\chi(M) = 2 - 2\,\text{genus}(M) - \#(\text{components of } \partial M) = 1 - 2\text{genus}(M),$$

we deduce that $\text{genus}(M) = 0$ and $\int_M K dM < 4\pi$. This proves that M is a topological disk and $N(M) \neq \mathbb{S}^2$. Since $\mathbf{n}$ takes each value of $\mathbb{S}^1 \times \{0\}$ exactly once and $N \circ \alpha \neq (0, 0, \pm 1)$, $N \circ \alpha$ is a curve that intersects each longitude of $\mathbb{S}^2$ exactly once and it does not pass through the two poles of $\mathbb{S}^2$. Thus $N_{|\Gamma}$ is bijective and $N(\Gamma)$ separates $\mathbb{S}^2$ into two simply connected components. Since N is an open map ($K > 0$), N is a diffeomorphism. This implies that N is a diffeomorphism between $int(M)$ and $\mathbb{S}^2 - N(\Gamma)$. As N does not assume both poles of $\mathbb{S}^2$, this means that the height function with respect to P has exactly one critical point. In particular, M lies on one side of P.

Let M be intersected by horizontal planes. Since N is one-to-one and the height function has one critical point, the previous argument proves that all level curves

must be strictly convex. This implies that M is embedded because it is foliated by convex curves in parallel planes. □

The Alexandrov method also applies under other types of assumptions on H, for example, when the mean curvature depends on a spatial coordinate. In order to simplify the statement of results, we focus only on the case where the boundary is a circle [Wen80a].

Corollary 4.1.10 *Let M be a compact embedded surface spanning a circle Γ contained in the plane P with equation $z = 0$. Assume the mean curvature function of M depends on the z-coordinate. If M lies on one side of P, then M is a surface of revolution whose axis is perpendicular to P. Moreover, each transversal intersection of a horizontal plane with M is a simple closed curve.*

Proof The proof is similar to that of Theorem 4.1.1 and Corollary 4.1.3 because if $M_{t_1}^*$ and $M_{t_1}^-$ touch tangentially at a point p, the mean curvature H of both surfaces agree at p since H depends only on the z-coordinate and the reflection through Π_t leaves this coordinate fixed. We indicate that the last statement is a consequence of the fact that each plane containing the axis of revolution separates M into two graphs on this plane. □

An application of this corollary is when H satisfies

$$H(x, y, z) = \kappa z + \mu, \quad \kappa, \mu \in \mathbb{R}. \tag{4.2}$$

In this case, the surface is a mathematical model of a liquid drop that rests (or hangs) on a plane in presence of gravity. Rotational surfaces are described by an ordinary differential equation. In fact, if the profile curve is $r \longmapsto (r, 0, u(r))$ and the rotational axis is the z-axis, then (4.2) is expressed as

$$\left(r \frac{u'(r)}{\sqrt{1 + u'(r)^2}} \right)' = 2r\big(\kappa u(r) + \mu\big).$$

A description of such surfaces appears in Finn's book [Fin86].

Remark 4.1.11 We call attention to the fact that there do not exist closed embedded surfaces satisfying $H(x, y, z) = \kappa z + \mu$, where $\kappa, \mu \in \mathbb{R}$, unless $\kappa = 0$, and in this case the surface is the sphere. Indeed, if M is a such surface, denote by W the domain bounded by the surface and let $V > 0$ be its volume. Consider in $\mathbb{R}^3$ the vector field $Y(x, y, z) = ze_3$. Then $\mathrm{Div}(Y) = 1$ and the divergence theorem implies

$$V = \int_W \mathrm{Div}(Y) = -\int_M z\langle N, e_3\rangle \, dM. \tag{4.3}$$

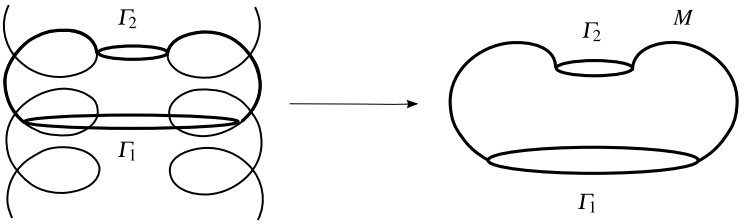

Fig. 4.4 *On the left* we fix two circles Γ_1 and Γ_2 in a nodoid. *On the right*, we extract the piece M of the nodoid bounded by $\Gamma_1 \cup \Gamma_2$. This embedded surface is not included in the slab determined by the planes containing Γ_1 and Γ_2

As M is closed, the divergence theorem also gives $\int_M \langle N, a\rangle \, dM = 0$ for all $a \in \mathbb{R}^3$. Integrating (2.22), using that $\partial M = \emptyset$ and (4.3), we conclude

$$0 = \int_M (\kappa z + \mu)\langle N, e_3\rangle \, dM = -\kappa V + \mu \int_M \langle N, e_3\rangle \, dM = -\kappa V.$$

Thus we derive $\kappa = 0$.

On the other hand, Theorem 4.1.1 extends to the case where the boundary is formed by two curves in parallel planes. We only state the result for two coaxial circles [Lop97a].

Corollary 4.1.12 *Consider two coaxial circles Γ_1 and Γ_2 in parallel planes P_1 and P_2, with $\Gamma_i \subset P_i$. Assume that M is a compact embedded cmc surface in $\mathbb{R}^3$ spanning $\Gamma_1 \cup \Gamma_2$. If int(M) lies in the slab determined by P_1 and P_2, then M is a surface of revolution.*

Proof By attaching the two planar disks $\Omega_i \subset P_i$ bounded by Γ_i, $i = 1, 2$, $M \cup \Omega_1 \cup \Omega_2$ is a closed embedded surface defining a bounded 3-domain $W \subset \mathbb{R}^3$. Now we apply the Alexandrov reflection method as in Theorem 4.1.1. Since $int(M)$ is included between the planes P_1 and P_2, the proof of Theorem 4.1.1 implies that if $t_1 > 0$ in claim 2, the point p where $M^*_{t_1}$ and $M^-_{t_1}$ touch is not a boundary point of M and thus, $M^*_{t_1}$ and $M^-_{t_1}$ are tangent at p. This gives that Π_{t_1} is a plane of symmetry, a contradiction. We conclude that $t_1 = 0$ and Π_0 is a plane of symmetry of M. By the same argument for any parallel direction to P_i, we conclude that M is rotationally symmetric about any straight line orthogonal to P_i. □

From this result, one expects that an embedded cmc surface bounded by two coaxial circles must be contained in the slab determined by the boundary planes. However there are embedded pieces of nodoids spanning two coaxial circles that are not included in the corresponding slab, as depicted in Fig. 4.4. Nevertheless we can ask if, under such assumptions, the surface must be rotationally symmetric.

When $H = 0$, we drop the assumption that the surface lies in the slab because of Theorem 3.3.1. We also weaken the assumption of embeddedness to mere immer-

sion since the choice of the orientation on a minimal surface does not affect the sign of the mean curvature, say, zero in this case.

Corollary 4.1.13 *The only compact minimal surface immersed in $\mathbb{R}^3$ bounded by two coaxial circles in parallel planes is a piece of a catenoid.*

Proof Let M be a minimal surface spanning two coaxial circles $\Gamma_1 \cup \Gamma_2$ in parallel planes $P_1 \cup P_2$. Without loss of generality, assume that $P_1 = \mathbb{R}^2$ and let $\Omega \subset \mathbb{R}^2$ be the round disk bounded by Γ_1. By the convex hull property of minimal surfaces, M lies in the slab determined by P_1 and P_2 and we have $int(M) \subset \Omega \times \mathbb{R}$. The proof follows the same argument as in Theorem 4.1.1 and Corollary 4.1.12. Comparing with the proof of Theorem 4.1.1 (and using the same notation), the difference now is the following. We begin with planes Π_t approaching from infinity until the position $t = t_0$. In claim 1, we have that M_t^+ is a graph on Π_t for $t \in (t_0 - \varepsilon, t_0)$, for some $\varepsilon > 0$. Next, we move Π_t to the left, reflecting M_t^+. The number t_1 in (4.1) is now defined as

$$t_1 = \inf\{t \in \mathbb{R} : M_s^* \text{ is a graph on } \Pi_s, \forall s \in (t, t_0)\}.$$

The proof of claim 2, that is, the proof that $t_1 > 0$ is not possible, is similar since there are neither interior points nor boundary points where $M_{t_1}^*$ is tangent to $M_{t_1}^-$ and $M_{t_1}^*$ lies on one side of $M_{t_1}^-$. The rest of proof is the same, proving that Π_0 is a plane of symmetry of M. Following this argument with all horizontal directions, the surface M is a rotational surface and thus, a piece of a catenoid by the classification of cmc surfaces of revolution [Del41]. □

The study of the shape of a minimal surface spanning two or more curves presents interesting questions. We focus on the case where Γ are two simple closed curves in parallel planes. When $\Gamma = \Gamma_1 \cup \Gamma_2$ are two circles, a theorem of M. Shiffman proves that the intersection of the minimal surface with each parallel plane to the boundary planes is a circle [Shi56]. If Γ_1 and Γ_2 are coaxial, the minimal surface is a piece of a catenoid by the above corollary; if not, the surface is one of the classical examples discovered by B. Riemann [Rie98]. The techniques employed in this context use complex analysis since the Gauss map of a minimal surface is conformal. On the other hand, if Γ_i are convex curves and the surface is a topological annulus, then the intersection of the surface with each plane parallel to the boundary planes is a convex curve. In relation to this result, W.H. Meeks III conjecturedthat a compact minimal surface spanning two convex curves in parallel planes must have the topology of an annulus [Mee78]. As a consequence of the proof of Corollary 4.1.13, we have a partial answer to this conjecture in the next result [Sch83]:

Corollary 4.1.14 *Let Γ_1 and Γ_2 be two simple closed curves in parallel planes P_1 and P_2 respectively. Assume that there exist two planes Π_1 and Π_2 orthogonal to P_i such that Π_i are planes of symmetry of $\Gamma_1 \cup \Gamma_2$ and Π_i separates $\Gamma_1 \cup \Gamma_2$ into two graphs on Π_i, $i = 1, 2$. If M is a minimal surface immersed in $\mathbb{R}^3$ with $\partial M = \Gamma_1 \cup \Gamma_2$, then M is embedded and has the topology of an annulus.*

Proof The reflection process implies that M is invariant under reflections through Π_1 and Π_2 and Π_i separates M_i into graphs over Π_i, $i = 1, 2$. Consider any plane P between P_1 and P_2. Then the curve $\gamma = M \cap P$, possibly disconnected, is invariant under reflections across the lines $L_i = \Pi_i \cap P$, $i = 1, 2$ and each piece of $\gamma \setminus (\gamma \cap L_i)$ is a graph on L_i. Then it is immediate that γ is connected, indeed, it is a simple closed curve. Since this hold for any such plane P, it follows that M is embedded and it is a topological annulus. □

In connection with this type of problem for minimal surfaces, the tangency principle allows us to obtain some results on the non-existence of minimal surfaces spanning two curves in parallel planes. The following is due to W. Rossman [Rsm98]:

Theorem 4.1.15 *Let Γ_1 and Γ_2 be two simple closed curves in parallel planes P_1 and P_2 respectively, and let $h = dist(P_1, P_2)$. Assume one of the following conditions*:

1. *There exists a plane Π orthogonal to P_i separating Γ_1 and Γ_2.*
2. *$\Gamma_1 \cup \Gamma_2$ lies in a slab perpendicular to P_i of width $w < h$.*
3. *The diameters d_i of Γ_i satisfy $h > \max\{d_1, d_2\}$.*

Then there does not exist a compact minimal surface immersed in $\mathbb{R}^3$ spanning $\Gamma_1 \cup \Gamma_2$.

The first item proves that if Γ_i are convex curves, the existence of a minimal surface spanning $\Gamma_1 \cup \Gamma_2$ implies that the orthogonal projection of Γ_1 on P_2 must overlap Γ_2. A proof of this fact for the Riemann examples appears in [Nit89, §96]. On the other hand, the last item improves a previous result of J.C.C. Nitsche who proved $h > (3/2)\max\{d_1, d_2\}$ assuming that the topology of the surface is an annulus [Nit64].

Proof Without loss of generality, assume that P_1 and P_2 are horizontal planes. With the aim of obtaining a contradiction we let M be a minimal surface with boundary $\Gamma_1 \cup \Gamma_2$. We know by Theorem 3.3.1 that $int(M)$ lies in the open slab determined by P_1 and P_2.

1. Assume that the vertical plane Π with equation $x = 0$ separates Γ_1 from Γ_2. Let M^* denote the reflection of M through Π, obtaining a minimal surface whose boundary is the reflection of Γ_i about Π. Move M^* in the direction of the y-axis until M^* does not intersect M and then translate M^* back until the first time that M^* touches M. Since the contact cannot occur at a boundary point, the point is interior in both surfaces. The tangency principle yields a contradiction.
2. Suppose that the two vertical planes are parallel to the plane Π with equation $x = 0$ and that Π is a plane equidistant from these planes. In addition, suppose that P_1 (resp. P_2) is the plane with equation $z = h/2$ (resp. $z = -h/2$). Let $\tilde{M}$ be the rotation by angle $\pi/2$ of M about the x-axis. Translate $\tilde{M}$ in the x-direction until it is disjoint from M. Then, move it back until the first intersection point with M.

The condition $w < h$ prohibits a contact between M and $\tilde{M}$ at a boundary point. Thus the contact must occur at an interior point, again yielding a contradiction by the tangency principle.

3. This is a direct consequence of the preceding item □

Returning to the statement of Theorem 4.1.3, the curve Γ is prescribed. The Alexandrov method can be applied under other boundary conditions, for example, by a capillary condition where we do not prescribe the boundary curve but the contact angle (see [Fin86], and also [Cho02, Par05, Pyo10]).

Theorem 4.1.16 *Let M be a compact embedded cmc surface in $\mathbb{R}^3$ with planar boundary Γ such that $int(M)$ lies on one side of the boundary plane P. If the angle between M and P along Γ is constant, then M is a spherical cap.*

Proof Following the proof of Theorem 4.1.1, we fix a horizontal direction $v \in \mathbb{R}^3$ and let Π_t be the 1-parameter family of parallel planes orthogonal to v that foliate the ambient space $\mathbb{R}^3$. Assume $v = (0, 1, 0)$ and that Π_t is the plane with equation $y = t$. We approach from $t = +\infty$ with planes Π_t until the first intersection with M. Next, we move to the left, reflecting the right part of the surface. Let t_1 be the number defined in (4.1). There are similar cases of contact points.

1. There is a point $p \in int(M^*_{t_1}) \cap int(M^-_{t_1})$ or $p \in (\partial M^*_{t_1} \cap \partial M^-_{t_1}) \setminus \Gamma$. The same argument proves that Π_{t_1} is a plane of symmetry of M.
2. At t_1, $M^*_{t_1}$ touches $M^-_{t_1}$ at a point $p \in \Gamma$. We distinguish two cases.
   a. $p \notin \Pi_{t_1}$. Since the angle between M and P along Γ is constant, $M^*_{t_1}$ is tangent to $M^-_{t_1}$ at p and $M^*_{t_1} \geq M^-_{t_1}$ around p. Observe that $M^*_{t_1}$ and $M^-_{t_1}$ are graphs around p on a domain of the half plane $T_pM \cap \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$. The boundary version of the tangency principle says that both surfaces agree on a neighborhood of p. By connectedness, Π_{t_1} is a plane of symmetry.
   b. $p \in \Pi_{t_1}$. In this case $M^*_{t_1}$ and $M^-_{t_1}$ are tangent at p again, but now the surfaces are locally graphs defined on a quadrant of T_pM, more precisely, on $(T_pM)^-_{t_1} \cap \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$. In this context, there is a version of the Hopf maximum principle that applies to this type of domain, called *Serrin's corner lemma*: see [GNN79, Mcc02, Ser71]. This proves that Π_{t_1} is a plane of symmetry.

Summarizing, we have proved that given any horizontal direction $v \in \mathbb{R}^3$, there exists a plane orthogonal to v, such that M is symmetric under reflections about this plane. Since this holds for all horizontal directions v, the surface must be a surface of revolution. As the surface is compact and the boundary is a simple closed plane curve, it follows that M is a spherical cap and the boundary Γ is a circle. □

The assumptions in Theorem 4.1.16 are both of Dirichlet and Neumann type because we impose that the boundary is planar (Dirichlet) and the contact angle is constant (Neumann). In such case we say that the problem is *overdetermined*.

In fact, Theorem 4.1.16 is a generalization of a theorem of J. Serrin for graphs which, in our context, asserts that given a solution of the constant mean curvature equation (3.3) on a domain Ω whose graph has planar boundary curve and the angle with the boundary plane is constant along $\partial\Omega$, then Ω must be a round disk and the surface is a planar disk or a spherical cap [Ser71].

Remark 4.1.17 The Alexandrov reflection method can be extended to other contexts, for example to surfaces in $\mathbb{R}^3$ with positive constant Gauss curvature. We have seen in Lemma 4.1.9 that if M a compact embedded surface with constant Gauss curvature K and the boundary ∂M is contained in a plane P, then $K > 0$ and M lies on a side of P. The fact that K is positive means that when writing the surface locally as the graph of a function $z = u(x, y)$, the function u satisfies an elliptic partial differential equation. Then the tangency principle can be formulated for this type of surface [Hop83] and the reflection method tells us that the surface inherits all symmetries of its boundary. See [AM02] for a generalization for hypersurfaces in $\mathbb{R}^{n+1}$ with spherical boundary that have a positive and constant r-mean curvature H_r, $n \geq 2$.

4.2 The Alexandrov Method with Horizontal Planes

Consider a compact embedded cmc surface spanning a curve included in a plane P. In the preceding section, we have used the Alexandrov method with planes orthogonal to P in order to obtain results of symmetry. However, we can also take planes parallel to P. The next theorem uses the Alexandrov reflection method with planes parallel to the plane containing the boundary. This was used by R. Schoen in [Sch83] to prove that under certain conditions, a minimal surface is a graph (see Corollary 4.2.4 below). The next theorem extends this result and it tells us that knowing the behavior of the surface around its boundary, we can control the shape of the surface [Lop97b].

Theorem 4.2.1 *Let M be a compact embedded cmc surface of $\mathbb{R}^3$ spanning a simple closed curve contained in a plane P. Let $\Omega \subset P$ be the domain bounded by ∂M and assume that M lies on one side of the plane P. If a neighborhood of ∂M in M is a graph over Ω, then M is a graph on Ω.*

Proof There is no loss of generality in assuming that P is the plane $z = 0$ and that M lies in the upper half space $z > 0$. The surface $M \cup (\partial\Omega \times (-\infty, 0))$ separates $\mathbb{R}^3$ into two connected components. Let W be the component containing Ω and let N be the orientation on M that points to W. We introduce the following notation (similar to that used in the proof of Theorem 4.1.1). For $t \in \mathbb{R}$, let P_t be the plane of equation $z = t$, $M_t^+ = M \cap \{(x, y, z) \in \mathbb{R}^3 : z \geq t\}$, $M_t^- = M \cap \{(x, y, z) \in \mathbb{R}^3 : z \leq t\}$ and denote by M_t^* the reflection of M_t^+ about P_t.

Let $t > 0$ be sufficiently large so that $M \cap P_t = \emptyset$. Decrease t until P_t first contacts M at $t = t_0$. As M lies above P_0 around ∂M, we have $t_0 > 0$. Since M is

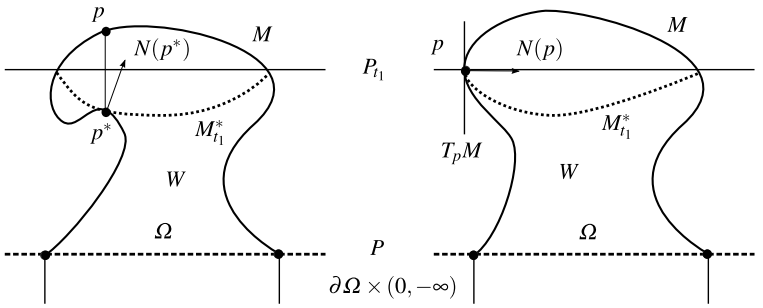

Fig. 4.5 Proof of Theorem 4.2.1. Both cases of tangent points are forbidden by the tangency principle

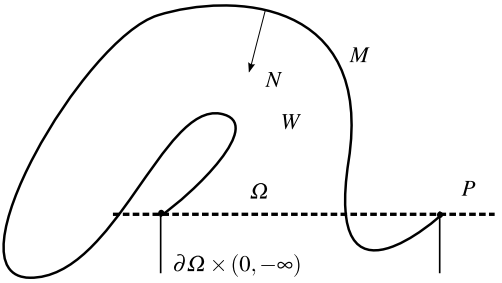

Fig. 4.6 The surface M does not exist by Corollary 4.2.2: a compact embedded cmc surface M with $\partial M = \partial\Omega$ that is a graph on Ω around $\partial\Omega$ and $M \cap (\Omega \times (-\infty, 0)) = \emptyset$

embedded, for some $\varepsilon > 0$, M_t^* lies inside $\overline{W}$ for each $t \in (t_0 - \varepsilon, t_0)$. Now we decrease the value of t, say, $t \searrow 0$, and as in Theorem 4.1.1, define

$$t_1 = \inf\{t \in \mathbb{R} : M_s^* \setminus \partial M_s^* \subset W,\, M_s^* \text{ is a graph on } \Pi_s,\, \forall s \in (t, t_0)\}.$$

As ∂M is included in P, we have $t_1 \geq 0$. If $t_1 = 0$, then M is a graph on Ω and this proves the result.

Aiming to arrive at a contradiction we suppose, on the contrary, that $t_1 > 0$. Note that there cannot exist a point $p \in int(M_{t_1}^+)$ such that its reflection p^* through P_{t_1} belongs to ∂M. This is because in the reflection method, the (vertical) open segment joining p and p^* is entirely included in W but, in contrast, and because M is a graph of Ω around ∂M, there is a $\delta > 0$ such that $(\partial M \times (0, \delta)) \cap W = \emptyset$.

Therefore, there are two possibilities concerning the point p^*: either $p^* \in int(M_{t_1}^*) \cap int(M_{t_1}^-)$ or $p \in \partial M_{t_1}^* \cap \partial M_{t_1}^-$. See Fig. 4.5. In both cases, $M_{t_1}^* \geq M_{t_1}^-$ around p^*. The tangency principle implies that $M_{t_1}^* = M_{t_1}^-$ in an open set around p^* and by connectedness, $M_{t_1}^* = M_{t_1}^-$. This proves that P_{t_1} is a plane of symmetry of the surface M contradicting that ∂M lies below P_{t_1}. □

In fact, we can extend the result even if ∂M is not a plane curve.

Corollary 4.2.2 *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain and let Γ be a simple closed curve contained in $\partial\Omega \times \mathbb{R}$ that is a graph on $\partial\Omega$. Let M be a*

compact embedded cmc surface in $\mathbb{R}^3$ *with boundary* Γ *such that int*(M) *does not intersect one of the two connected components of* $(\partial\Omega \times \mathbb{R}) \setminus \Gamma$. *If a neighborhood of* ∂M *in* M *is a graph on* Ω, *then* M *is a graph on* Ω.

When we say that Γ is a graph on $\partial\Omega$, we mean that Γ is the graph of a function $\varphi \in C^\infty(\partial\Omega)$. In particular, the tangent vector of Γ is never vertical. In the terminology of [Sch83], we say that Γ is a graph of bounded slope.

Proof Assume that Γ is the graph of a function φ defined on $\partial\Omega$. Then the two connected components that Γ defines on $\partial\Omega \times \mathbb{R}$ are $C^+ = \{(q, z) : q \in \partial\Omega, z > \varphi(q)\}$ and $C^- = \{(q, z) : q \in \partial\Omega, z < \varphi(q)\}$. Without loss of generality, assume that $M \cap C^- = \emptyset$. Consider the embedded surface $M \cup C^-$, which separates $\mathbb{R}^3$ into two connected components by the Alexander duality [GH81]. If $z_0 = \min\{z(p) : p \in M\}$, let W be the connected component that contains $\Omega \times (-\infty, z_0)$. Consider N pointing to W.

The proof works as in Theorem 4.2.1 by reflections about P_t approaching from $t = +\infty$. If there are no contact points between M_t^* and M_t^-, then M is a graph on Ω and the result is proved.

Assume to the contrary that at a time t_1, there exists a $p \in M_{t_1}^+$ such that $p^* \in M_{t_1}^* \cap M_{t_1}^-$ and $M_{t_1}^* \geq M_{t_1}^-$ around p^*. Since M is locally a graph around ∂M, it is not possible that $p \in M_{t_1}^+$, $p^* \in \partial M$, $p \neq p^*$ because the segment (p, p^*) would be included in W. In the cases $p^* \in int(M_{t_1}^*) \cap int(M_{t_1}^-)$ or $p = p^* = (\partial M_{t_1}^* \cap \partial M_{t_1}^-) \setminus \Gamma$, the tangency principle implies that P_{t_1} is a plane of symmetry of M. In particular, it is a plane of symmetry of Γ. This implies that Γ is a plane curve with $\Gamma \subset P_{t_1}$, a contradiction. □

If the boundary Γ coincides with $\partial\Omega$, the assumption of the corollary allows that M intersects the plane $z = 0$: see Fig. 4.6. A consequence of Corollary 4.2.2 is:

Corollary 4.2.3 *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected bounded domain and let* $\Gamma \subset \partial\Omega \times \mathbb{R}$ *be a simple closed curve that is a graph on* $\partial\Omega$. *If* M *is a compact embedded cmc surface spanning* Γ *and int*$(M) \subset \Omega \times \mathbb{R}$, *then* M *is a graph on* Ω.

If $H = 0$, we have the following result [Sch83]:

Corollary 4.2.4 *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded convex domain and let* $\Gamma \subset \partial\Omega \times \mathbb{R}$ *be a simple closed curve that is a graph on* $\partial\Omega$. *If* M *is a compact minimal surface immersed in* $\mathbb{R}^3$ *spanning* Γ, *then* M *is a graph on* Ω.

Proof By Theorem 3.3.1, $int(M) \subset \Omega \times \mathbb{R}$. We use the reflection technique with horizontal planes P_t in a similar way as in Corollary 4.1.13. Since $\Gamma \subset \partial\Omega \times \mathbb{R}$, if there is a tangent point between M_t^* and M_t^-, this point must interior which is not possible by the tangency principle. Then we continue reflecting and no contact points appear, proving the result. □

The Alexandrov method allows us to estimate the height of a compact embedded cmc surface [Mee88]. For a solution of the Plateau problem, this estimatewas obtained by J. Serrin in [Ser69a].

Corollary 4.2.5 *Let $H \neq 0$. Let $M \subset \mathbb{R}^3$ be a compact embedded H-surface with boundary ∂M contained in a plane P. Then the height of M with respect to P is less than or equal to $2/|H|$.*

Proof Assume P is the plane $z = 0$ and let $\tilde{M}$ be the closure of a connected component of $M \setminus (M \cap P)$. Denote by h the height of $\tilde{M}$ with respect to P and assume that $\tilde{M}$ lies above P. Then $\partial \tilde{M} \subset P$ and $\tilde{M}$ together with some domains of P defines a bounded 3-domain $W \subset \mathbb{R}^3$. We use the Alexandrov method with horizontal planes as in Theorem 4.2.1. Approaching from $t = +\infty$ in the reflection process, we continue until P_t first touches M at $t = t_0$. Here $t_0 = h$. Next, we continue with the process in such way that $\tilde{M}_t^* \subset \overline{W}$. This occurs at least for each $t \in (h/2, h)$ because $\tilde{M}_t^* \cap P = \emptyset$. In addition, for these values of t, each surface $\tilde{M}_t^*$ is a graph on P_t. By Theorem 2.3.5 the height of this graph, say, $h - t$, is less than $1/|H|$. Taking $t \to h/2$, we conclude $h - h/2 \leq 1/|H|$, proving the result. □

The hypothesis on embeddedness is necessary because of Kapouleas' examples [Kap91]. Recall that given a circle Γ and $H \in \mathbb{R}$, Kapouleas constructed immersed H-surfaces spanning Γ with arbitrarily large heights.

If the boundary is not planar, we have:

Corollary 4.2.6 *Let M be a compact embedded H-surface in $\mathbb{R}^3$ with $H \neq 0$. Assume that ∂M is included in the closure of a ball $B(p, r)$, $r > 0$. Then M is included in the closure of $B(p, r + 2/|H|)$.*

Proof Let P be a tangent plane at a point $S(p, r)$. Then the part of M on the side of P that does not contain ∂M is an embedded H-surface with planar boundary and we apply the previous corollary. □

This result indicates that any compact embedded H-surface with boundary Γ is included in a compact subset of $\mathbb{R}^3$ whose size depends only on Γ and H. Note that if $H = 0$, then M is included in $B(p, r)$ by Theorem 3.3.1. In order to emphasize this fact, we state the result as follows:

Corollary 4.2.7 *Let $\Gamma \subset \mathbb{R}^3$ be a simple closed curve and $H \in \mathbb{R}$. Then there exists a constant $C(\Gamma, H)$ depending only on Γ and H such that*

$$\max_{p \in M} |p| \leq C(\Gamma, H)$$

for all compact embedded H-surface in $\mathbb{R}^3$ spanning Γ.

The same technique with horizontal planes applies if the boundary curve is formed by two curves in parallel planes [Lop97a].

Theorem 4.2.8 *Let P_1 and P_2 be two horizontal planes, where P_i has equation $z = c_i$, $i = 1, 2$. Let Γ_1, Γ_2 be two simple closed curves with $\Gamma_i \subset P_i$, $i = 1, 2$, such that Γ_2 is the orthogonal projection of Γ_1 on P_2. Let M be a compact embedded cmc surface spanning $\Gamma_1 \cup \Gamma_2$. If $M \cap (\Gamma_1 \times \mathbb{R}) = \Gamma_1 \cup \Gamma_2$, then M is symmetric with respect to the plane $z = (c_1 + c_2)/2$. Moreover, this plane separates M into two graphs.*

Proof We attach the cylinder $\Gamma_1 \times [c_1, c_2]$ to M obtaining a closed embedded surface which it is not smooth along ∂M. We apply the Alexandrov method by the planes P_t as in Theorem 4.2.1 approaching from $t = +\infty$. The assumptions ensure that the contact point p^* between $M_{t_1}^*$ and $M_{t_1}^-$ does not belong to $\Gamma_1 \cup \Gamma_2$. The reflection method implies that the plane P_{t_1} is a plane of symmetry of M, in particular, of $\Gamma_1 \cup \Gamma_2$. Therefore $t_1 = (c_1 + c_2)/2$. Moreover, M_t^+ is a graph on P_t for each $t > t_1$, implying that $M \cap \{(x, y, z) \in \mathbb{R}^3 : z > t_1\}$ and $M \cap \{(x, y, z) \in \mathbb{R}^3 : z < t_1\}$ are graphs on P_{t_1}. □

4.3 Embedded Surfaces with Boundary in a Sphere

Consider a compact embedded cmc surface with boundary included in a sphere. For simplicity, we assume that this sphere is centered at the origin and of radius $\rho > 0$ and we denote it by $\mathbb{S}^2_\rho$. Denote by $\mathbb{S}^2_{\rho,+} = \{p \in \mathbb{S}^2_\rho : z(p) > 0\}$ the open upper hemisphere and the lower hemisphere by $\mathbb{S}^2_{\rho,-} = \mathbb{S}^2_\rho \setminus \overline{\mathbb{S}^2_{\rho,+}}$. A simple closed curve Γ included in $\mathbb{S}^2_{\rho,+}$ separates $\mathbb{S}^2_{\rho,+}$ into two connected components. We call *the domain bounded by* Γ the component of $\mathbb{S}^2_{\rho,+} \setminus \Gamma$ whose boundary is Γ (the boundary of the other connected component is $\partial \mathbb{S}^2_{\rho,+} \cup \Gamma$).

In this section we study when the surface lies on one side of $\mathbb{S}^2_\rho$. In the present case, each side corresponds with B_ρ or E_ρ, where

$$B_\rho = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < \rho^2\right\},$$
$$E_\rho = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 > \rho^2\right\}.$$

The next result is analogous to Theorem 4.1.1 [LP13].

Theorem 4.3.1 *Let Γ be a simple closed curve included in an open hemisphere of $\mathbb{S}^2_\rho$. Assume that Π is a vector plane through the pole of the hemisphere, which is a plane of symmetry of Γ, and assume that Π divides Γ into two graphs defined on $\Pi \cap \mathbb{S}^2_\rho$. Suppose that M is a compact embedded H-surface spanning Γ and $|H| \geq 1/\rho$. If $\text{int}(M) \subset E_\rho$, then M is symmetric with respect to the plane Π.*

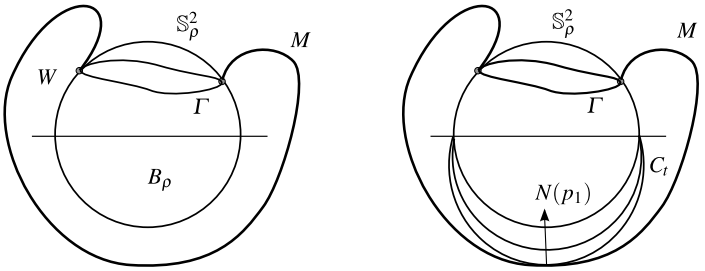

Fig. 4.7 *Left*: the case where B_ρ is included in the domain W. *Right*: a contradiction is obtained by comparing M with the spherical caps C_t

Here we say that Π divides Γ into two graphs on $\Pi \cap \mathbb{S}^2_\rho$ if any circle of $\mathbb{S}^2_\rho$ contained in a plane orthogonal to Π and parallel to the axis through the pole intersects each component of $\Gamma \setminus (\Gamma \cap \Pi)$ at most once.

Proof We write $\Gamma \setminus (\Gamma \cap \Pi) = \Gamma^- \cup \Gamma^+$, where Γ^- is the symmetry of Γ about Π and, in addition, Γ^- and Γ^+ are graphs on $\Pi \cap \mathbb{S}^2_\rho$. After a rigid motion, assume that the open hemisphere is $\mathbb{S}^2_{\rho,+}$ and the plane Π is $x = 0$. Let $\Omega \subset \mathbb{S}^2_{\rho,+}$ be the domain bounded by Γ and let $T = M \cup \overline{\Omega}$. Observe that T is a connected compact topological surface of $\mathbb{R}^3$ without boundary. Therefore, T bounds a domain $W \subset \mathbb{R}^3$ by the Alexander duality. We orient M by the unit normal vector field N pointing towards W. With this orientation, the mean curvature H is positive: it suffices to compare M with the tangent plane at the farthest point $p_0 \in M$ from the origin.

Since $T \cap B_\rho = \emptyset$, there are two possibilities concerning the domain W, namely, $B_\rho \subset \mathbb{R}^3 \setminus \overline{W}$ or $B_\rho \subset W$.

We rule out the second case. See Fig. 4.7, left. If this occurs, take the 1-parameter family of spherical caps C_t contained in $\overline{E_\rho} \cap \{(x, y, z) \in \mathbb{R}^3 : z \leq 0\}$, $t \in [\rho, \infty)$, whose boundary is the equator $\mathbb{S}^2_\rho \cap \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and where t indicates the radius of C_t. Fix the orientation on C_t pointing towards the center of C_t. For $t = \rho$, C_ρ is the closed hemisphere $\overline{\mathbb{S}^2_{\rho,-}}$. Starting from $t = \rho$, we increase the radius of C_t. For values of t close to $t = \rho$, the spherical cap C_t is included in $\overline{W}$ by the property that $\Gamma \subset \mathbb{S}^2_{\rho,+}$. Since W is bounded, we increase t until C_t first intersects M at $t = t_1 > \rho$ as depicted in Fig. 4.7, right. This must occur at an interior point p_1 of both surfaces. The unit normal vectors of C_{t_1} and M agree at p_1 because both vectors point towards W. Now, $C_{t_1} \geq M$ around p_1 but the mean curvature of C_{t_1} is $1/t_1$ and $1/t_1 < 1/\rho \leq H$, contradicting the comparison principle.

As a consequence, the ball B_ρ is contained in $\mathbb{R}^3 \setminus \overline{W}$, or equivalently, $W \subset E_\rho$. Now we apply the Alexandrov reflection method in two steps. First, we consider the family of horizontal planes $\{P_t : t \in \mathbb{R}\}$ with equation $z = t$. Since W is a bounded set, let $t < 0$ be so that $W \subset \{(x, y, z) \in \mathbb{R}^3 : z > t\}$. Next, we increase t until P_t first contacts M at $t = t_1$: $P_t \cap M = \emptyset$ for $t < t_1$ and $P_{t_1} \cap M \neq \emptyset$. Then $t_1 \leq \bar{t} = \min\{z(q) : q \in \Gamma\}$ and $\bar{t} > 0$, where $z(q)$ denotes the z-coordinate of q. Next, we increase t and we reflect the part of M below P_t about the plane P_t. We introduce the

following notation. Set $M_t^+ = M \cap \{(x, y, z) \in \mathbb{R}^3 : z \geq t\}$, $M_t^- = M \cap \{(x, y, z) \in \mathbb{R}^3 : z \leq t\}$ and let M_t^* be the reflection of M_t^- through the plane P_t. The number t_1 satisfies $t_1 < \bar{t}$ or $t_1 = \bar{t}$.

Assume first that $t_1 < \bar{t}$, that is, P_{t_1} intersects M at an interior point of M. For some $\varepsilon > 0$, M_t^* is included in $\overline{W}$ for $t \in (t_1, t_1 + \varepsilon)$ and M_t^* does not make any tangential contact point with M_t^+. We keep increasing t and reflecting M_t^- until the first time t_2 such that $M_{t_2}^* \not\subset \overline{W}$, which exists by the boundedness of W. First, assume that $t_2 \leq 0$. Then $M_{t_2}^*$ touches ∂W at a point p_2. There are two possibilities, similar to those in Theorem 4.1.1:

1. There exists a $q_2 \in M_{t_2}^-$ such that $q_2 \neq p_2$ and $q_2^* = p_2$. In particular, the vertical open segment joining q_2 with p_2 lies in W. Then $p_2 \notin \Gamma$ because $B_\rho \cap W = \emptyset$. This implies that $p_2 \in int(M_{t_2}^+)$ and the tangency principle assures us that $M_{t_2}^+ = M_{t_2}^*$ in an open neighborhood of p_2.
2. The point p_2 is a common boundary point of $M_{t_2}^+$ and $M_{t_2}^*$. This means that $p_2 \in P_{t_2}$ and $p_2 \notin \Gamma$ again because $t_2 \leq 0 < \bar{t}$. The surfaces $M_{t_2}^+$ and $M_{t_2}^*$ are tangent at p_2 (in fact, the tangent plane T_pM is perpendicular to P_{t_2}) and the boundaries of $M_{t_2}^+$ and $M_{t_2}^*$ are tangent at p_2. In particular, the surfaces $M_{t_2}^+$ and $M_{t_2}^*$ are graphs in some domain of $T_pM \cap \{(x, y, z) \in \mathbb{R}^3 : z \geq t_2\}$. The boundary version of the tangency principle implies that $M_{t_2}^+ = M_{t_2}^*$ in an open neighborhood of p_2.

In both cases, denote by K^* the connected component of $M_{t_2}^*$ that contains p_2. Then K^* is the reflection about P_{t_2} of a certain connected component K of $M_{t_2}^-$. As $M_{t_2}^* \cap \Gamma = \emptyset$, we repeat the argument with the tangency principle and we derive that $K^* \cap M$ is an open and closed subset of K^*. Since K^* is connected, $K^* \cap M = K^*$, hence K^* is contained in M. Since $M_{t_2}^* \cap \Gamma = \emptyset$, it follows that K^* is included in $M \setminus \Gamma$. Therefore ∂K coincides with ∂K^*. This proves that $\overline{K \cup K^*}$ is a surface without boundary contained in M and this implies that it coincides with M, which contradicts the assumption that $\partial M \neq \emptyset$.

In view of the above reasoning, the case $t_2 \leq 0$ is not possible, and that the reflection process with horizontal planes P_t continues until position $t = 0$ and, furthermore, M_0^* lies in W. Now, we change the reflection process using a different uniparametric family of planes. For each $t \in [\pi/2, \pi]$, denote by Q_t the plane with equation $\cos(t)x + \sin(t)z = 0$ that is orthogonal to the vector $(\cos(t), 0, \sin(t))$. All planes Q_t have a common straight line L, namely, with equation $x = z = 0$. The plane $Q_{\pi/2}$ coincides with P_0 and Q_π coincides with Π. See Fig. 4.8. We introduce the following notation.

$$M_Q(t)^+ = \left\{(x, y, z) \in M : \cos(t)x + \sin(t)z \geq 0\right\}$$
$$M_Q(t)^- = \left\{(x, y, z) \in M : \cos(t)x + \sin(t)z \leq 0\right\}$$

and $M_Q(t)^*$ denotes the reflection of $M_Q(t)^-$ with respect to Q_t. Let Ψ_t denote the reflection through Q_t. In addition, let $\Gamma^+ = \Gamma \cap \{x > 0\}$ and $\Gamma^- = \Gamma \cap \{x < 0\}$.

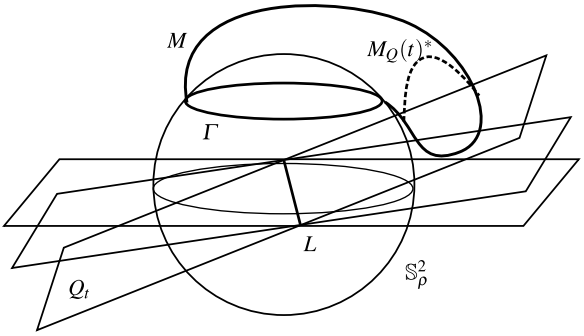

Fig. 4.8 Proof of Theorem 4.3.1. Reflections through the planes Q_t

The idea is to follow the Alexandrov reflection method by means of the planes Q_t starting at $t = \pi/2$ and showing that we reach the value $t = \pi$. We record some properties of these reflections.

1. Since Ψ_π is the reflection about the plane Π, we have $\Psi_\pi(\Gamma^+) = \Gamma^-$.
2. It is possible to continue with the Alexandrov process by replacing the family P_t by Q_t. Indeed, for $t = \pi/2$, the plane $Q_{\pi/2}$ is P_0, the surface M_0^* coincides with $M_Q(0)^*$ and thus, $M_Q(0)^*$ is included in $\overline{W}$.
3. The reflections about Q_t leave $\mathbb{S}^2_\rho$ invariant.
4. For each $t \in [\pi/2, \pi)$, if Q_t contains a point $p \in \Gamma$, then $p \in \Gamma^+$. Moreover, the set $\{\Psi_t(p) : t \in [\pi/2, \pi)\}$ describes an interval of a half circle in $\mathbb{S}^2_{\rho,+}$ which is contained in a plane orthogonal to Π parallel to the z-axis and intersecting Γ in exactly two points, namely, p and $\Psi_\pi(p)$. This is because Γ^+ and Γ^- are graphs on $\Pi \cap \mathbb{S}^2_\rho$.

We increase $t \nearrow \pi$ until the first time t_3 at which $M_Q(t_3)^*$ is no longer included in $\overline{W}$. We claim that $t_3 = \pi$. Assume $t_3 < \pi$. Let $p_3 \in M_Q(t_3)^+ \cap M_Q(t_3)^*$. There are two possibilities.

1. The point p_3 does not belong to Γ^-. Then $p_3 \in int(M_Q(t_3)^+) \cap int(M_Q(t_3)^*)$ or $p_3 \in \partial M_Q(t_3)^+ \cap \partial M_Q(t_3)^*$. An argument with the tangency principle, similar to that for the reflections with the planes P_t, says that Q_{t_3} is a plane of symmetry of M, in particular, of Γ: a contradiction because $t_3 < \pi$.
2. The point p_3 belongs to Γ^-. In this case, there would be a point $q_3 \in \Gamma^+$ such that $\Psi_{t_3}(q_3) = p_3$. But this is not possible because it can only occur if $t_3 = \pi$ by the symmetry of Γ about Π.

Thus $t_3 = \pi$. Having established this, there are now two possibilities. First suppose that $M_Q(\pi)^* \setminus \Gamma^-$ is completely included in W. In this case, we follow an analogous process of reflection about the planes Q_t with $t \in [\pi, 3\pi/2]$ starting from $t = 3\pi/2$ and letting $t \searrow \pi$. The same reasoning and the fact that $M_Q(\pi)^* \setminus \Gamma^- \subset W$ would imply the existence of $t_4 \in (\pi, 3\pi/2)$ such that Q_{t_4} is a plane of symmetry of M, which is a contradiction. Therefore, the only possibility is that $M_Q(\pi)^* \setminus \Gamma^-$ touches ∂W at a point, which necessarily belongs to $M_Q(\pi)^-$. The tangency principle implies that $Q_\pi = \Pi$, proving the theorem when $t_1 < \bar{t}$.

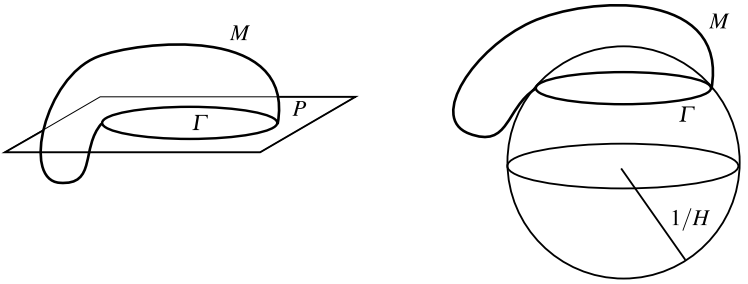

Fig. 4.9 *Left*: A possible compact embedded cmc surface M spanning a circle Γ. *Right*: The surface M is not possible by Corollary 4.3.2

The remaining possibility is if $t_1 = \bar{t}$, that is, by moving P_t upward, we do not intersect M until P_t arrives at the value $t = \bar{t}$, the lowest height of Γ. Then we move P_t back until the position $t = 0$ and we begin the reflection method using the planes Q_t as above. The difference now is that when we start, we have $Q_{\pi/2} \cap M = \emptyset$ and we arrive at the first point $t_1 \in (\pi/2, \pi)$ of contact between Q_{t_1} and M. Next, we continue with t and reflect about Q_t until the first time t_2 that $M_Q(t_2)^*$ leaves $\overline{W}$. Now the proof follows the same steps, proving the result. □

If Γ is a circle, we conclude

Corollary 4.3.2 *Let $H \neq 0$ and let Γ be a circle with radius $r > 0$, $r \leq 1/|H|$. Suppose that M is a compact embedded H-surface spanning Γ such that $int(M) \subset E_{1/|H|}$, where $\mathbb{S}^2_{1/|H|}$ is one of the two spheres of radius $1/|H|$ containing Γ. Then M is a spherical cap.*

Proof If $r = 1/|H|$, M is a hemisphere of radius $1/|H|$ by a result of Brito and Earp (see Corollary 5.1.11). But this hemisphere is not included in $E_{1/|H|}$ and hence this case is impossible. Therefore, $r < 1/|H|$. Let P be the plane through the center of $\mathbb{S}^2_{1/|H|}$ and parallel to the plane containing Γ. The plane P defines a unique open hemisphere containing Γ. We apply Theorem 4.3.1 for any plane of symmetry of Γ, concluding that M is invariant under rotations with respect to the straight line orthogonal to P containing the center of Γ. This shows that M is a surface of revolution and the only compact rotational non-zero cmc surface spanning a circle is a spherical cap. □

Corollary 4.3.2 states that the surface that appears in Fig. 4.9, right, is not possible.

In the proof of Theorem 4.3.1, note that the assumption $int(M) \subset E_\rho$ has been used before the Alexandrov method in order to prove that B_ρ is not included in W. Therefore, we drop the assumption on the value of the radius of the sphere as follows.

Corollary 4.3.3 *Let Γ be a simple closed curve included in an open hemisphere of $\mathbb{S}^2_\rho$ and assume that Π is a plane of symmetry of Γ that separates Γ into two graphs defined on $\Pi \cap \mathbb{S}^2_\rho$. Suppose that M is a compact embedded cmc surface spanning Γ and M satisfies one of the following two assumptions*:

1. *$int(M) \subset B_\rho$.*
2. *The bounded domain W defined by $M \cup \overline{\Omega}$ is included in E_ρ.*

Then M is symmetric with respect to the plane Π. If Γ is a circle, the surface is a spherical cap.

Proof By an isometry in $\mathbb{R}^3$, assume that $\Pi = \{x = 0\}$ and that the open hemisphere is $\mathbb{S}^2_{\rho,+}$. The Alexandrov method works as in Theorem 4.3.1 beginning with horizontal planes P_t. We only sketch the proof, highlighting any differences and only focussing on the case $int(M) \subset B_\rho$. We continue at least until $t = 0$ and no contact points appear between M_t^* and M_t^+: on the contrary, the tangency principle would give a plane of symmetry P_{t_2} for some $t_2 \leq 0$, which is not possible. In this part of the proof, we point out that a point of M_t^* cannot touch a point of Γ because the reflection of a point of $M \cap B_\rho$ about P_t with $t < 0$ must lie in B_ρ. Once arrived at $t = 0$, we use reflections through the planes Q_t, obtaining the result. □

We obtain another consequence of the proof of Theorem 4.3.1. We need the following result.

Proposition 4.3.4 *Let $\Gamma \subset \mathbb{S}^2_{\rho,+}$ be a simple closed curve. Let M be a compact embedded surface spanning Γ such that $int(M)$ is included in E_ρ or in B_ρ. Then there exists a subset $\Omega \subset \mathbb{S}^2_{\rho,+}$ such that $M \cup \Omega$ is a closed embedded surface of $\mathbb{R}^3$. In particular, $M \cup \Omega$ defines a bounded domain $W \subset \mathbb{R}^3$.*

Corollary 4.3.5 *Let $\Gamma \subset \mathbb{S}^2_{\rho,+}$ be a simple closed curve. Let M be a compact embedded H-surface spanning Γ with $|H| \geq 1/\rho$. If $int(M) \subset E_\rho$, then the domain W defined in Proposition 4.3.4 is included in E_ρ.*

Proof If $W \not\subset E_\rho$, then $B_\rho \subset W$. The first part of the proof of Theorem 4.3.1 holds proving that with the orientation pointing towards W the mean curvature H is positive. Now we use the argument used there by comparing with the same spherical caps C_t, obtaining a contradiction. □

We devote the remainder of this section to proving a result similar to the Koiso theorem (Theorem 3.1.11). A crucial fact in the proof of Theorem 3.1.11 was that the ambient space $\mathbb{R}^3$ is foliated by a family of parallel planes and this allowed us to sweep the space $\mathbb{R}^3$ with planes comparing the mean curvature of the surface with that of a plane.

The other umbilical surfaces of $\mathbb{R}^3$ are spheres. Fixing a point $p \in \mathbb{R}^3$, the class of spheres centered at p foliate the space, but in contrast to the planes, each sphere has a different (constant) mean curvature. In this context, a similar situation is the following [Lop97d]:

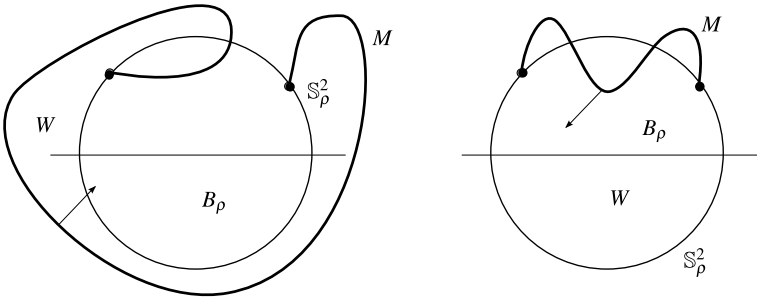

Fig. 4.10 The two possibilities of W in the proof of Theorem 4.3.6. *On the left*, W lies in E_ρ near $\mathbb{S}^2_{\rho,-}$. *On the right*, the domain W lies in B_ρ near $\mathbb{S}^2_{\rho,-}$. The first case is ruled out, as is proved in Fig. 4.11

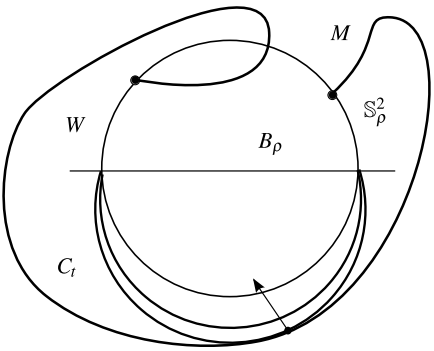

Fig. 4.11 Comparing M with a spherical cap spanning $\partial\mathbb{S}^2_-$ below the plane $z = 0$ obtained by inflating $\mathbb{S}^2_{\rho,-}$

Theorem 4.3.6 *Let $\Gamma \subset \mathbb{S}^2_{\rho,+}$ be a simple closed curve and denote by Ω the domain bounded by Γ. Let M be a compact embedded surface spanning Γ with mean curvature function H, $|H| \geq 1/\rho$. If $M \cap (\mathbb{S}^2_\rho \setminus \overline{\Omega}) = \emptyset$, then either M is $\overline{\Omega}$, or $int(M) \subset B_\rho$ or $int(M) \subset E_\rho$.*

Proof Suppose $M \neq \overline{\Omega}$. To prove the result, it suffices to show that $int(M) \cap \Omega = \emptyset$. Let us assume for contradiction that $int(M) \cap \Omega \neq \emptyset$. Without loss of generality, suppose that there are interior points of M with modulus less than ρ and points with modulus greater than or equal to ρ. Consider the closed embedded surface $M \cup (\mathbb{S}^2_\rho \setminus \overline{\Omega})$ and denote by $W \subset \mathbb{R}^3$ the 3-domain that it bounds. Choose on M the orientation N corresponding to $H > 0$. With a reasoning similar to the proof of Theorem 4.3.1, N points towards W. There are two possibilities as depicted in Fig. 4.10:

1. The domain W lies in E_ρ near $\mathbb{S}^2_{\rho,-}$.
2. The domain W lies in B_ρ near $\mathbb{S}^2_{\rho,-}$.

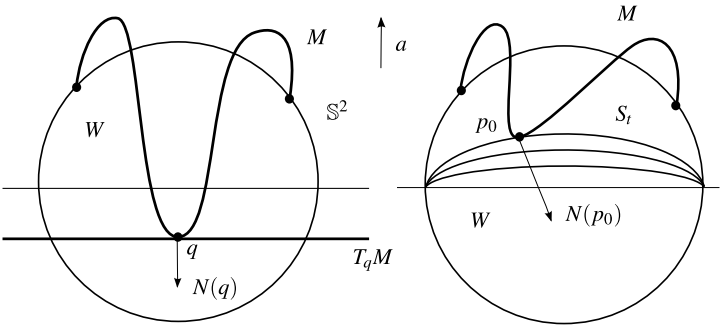

Fig. 4.12 Comparing M with horizontal planes and spherical caps S_t in the proof of Theorem 4.3.6

We rule out the first case. We inflate $\mathbb{S}^2_{\rho,-}$ fixing the boundary $\partial\mathbb{S}^2_{\rho,-}$ obtaining a 1-parameter family of large spherical caps C_t of radius $t \geq \rho$, spanning $\partial\mathbb{S}^2_{\rho,-}$ and below the plane $z = 0$: see Fig. 4.11.

At the beginning, $C_t \subset \overline{W}$ for $t > \rho$ and t close to the value ρ. Increasing the radius t and by the compactness of $\overline{W}$, there is a radius $t_0 > \rho$ at which C_{t_0} touches M the first time. Since this occurs at a point $p \in int(M)$, the comparison principle yields $1/t_0 \geq H(p)$, a contradiction because $H \geq 1/\rho$. We conclude that the domain W is included in B_ρ near to $\mathbb{S}^2_{\rho,-}$.

Now we claim that $M \cap B_\rho$ does not intersect the halfspace $\{(x, y, z) \in \mathbb{R}^3 : z \leq 0\}$. On the contrary, let $q = (q_1, q_2, q_3) \in M \cap B_\rho$, $q_3 \leq 0$, be the point where the third coordinate function z attains its minimum in the subset $M \cap B_\rho$. Then $N(q) = (0, 0, -1)$ because $N(q)$ points towards W. Compare M with the (horizontal) tangent plane T_qM. See Fig. 4.12, left. As $T_qM \geq M$ around q, the comparison principle gives $0 \geq H(q)$, a contradiction.

Therefore $M \cap B_\rho \subset \{p \in \mathbb{R}^3 : z > 0\}$. Consider the family of small spherical caps S_t, $t \in (0, 1/\rho]$ included in the halfspace $\{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ spanning the equator $\partial\mathbb{S}^2_{\rho,+}$ and with constant mean curvature $t > 0$ (with respect to the downward orientation). Since M is a compact surface and $M \cap B_\rho$ is included in the upper halfspace $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, there exists an $\varepsilon > 0$ such that $S_t \cap M = \emptyset$ for all $t \in (0, \varepsilon)$. Increase $t \to 1/\rho$ until S_t first intersects M at $t = t_0 > \varepsilon$. Note that $t_0 < 1/\rho$ because $M \cap B_\rho \neq \emptyset$. Then S_{t_0} and M are tangent at a point $p_0 \in int(M)$: see Fig. 4.12, right. Since $S_{t_0} \geq M$ around p_0, the comparison principle gives $1/\rho > t_0 \geq H(p_0)$, a contradiction. □

Chapter 5
The Flux Formula for Constant Mean Curvature Surfaces

In the preceding chapter we have seen that, under some conditions, the geometry of the boundary of a compact cmc surface is inherited by the surface. The problem that we treat in this chapter concerns the question of whether or not there are restrictions on the possible values H of mean curvatures for the existence of an H-surface spanning a given boundary curve Γ. We illustrate this problem with a circle Γ of radius r. The first example of an H-surface spanning Γ is the planar domain bounded by Γ, which corresponds with $H = 0$ and we continue with small spherical caps of mean curvature H where H varies from 0 to $1/r$. For $H = 1/r$, we reach the hemisphere bounded by Γ. Next, increasing the radius of the spheres, we obtain large spherical caps spanning Γ and the mean curvature H decreases to 0. Thus, the condition for the existence of spherical H-caps spanning Γ is that $|H| \leq 1/r$.

In general, the geometry of Γ imposes restrictions on the values of the mean curvatures H, indeed, we will see that not all values of H are admissible. This has been seen for H-graphs defined on a given bounded domain $\Omega \subset \mathbb{R}^2$. In such a case, $|H| \leq L(\partial\Omega)/(2a(\Omega))$.

In this chapter, we shall obtain a flux formula for a compact cmc surface M immersed in $\mathbb{R}^3$ that relates integrals of M and of its boundary ∂M, linking information between the geometry of M and ∂M. We will combine the flux formula with the tangency principle in order to control the shape of a cmc surface with boundary. For example, if the boundary is a circle and the surface is contained in a Euclidean ball or a cylinder whose size is related to the mean curvature of the surface, we shall deduce that the surface is spherical. Finally, and for an embedded surface with planar boundary Γ, we will prove that the surface lies on one side of the boundary plane knowing only that the surface is transverse to this plane along Γ.

5.1 The Flux Formula

Let us begin with the statement of the flux formula. For embedded surfaces, the formula was obtained by R. Kusner in [Kus88, Kus91].

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_5,
© Springer-Verlag Berlin Heidelberg 2013

Theorem 5.1.1 (Flux formula) *Let M be a compact surface and let $x : M \to \mathbb{R}^3$ be an immersion with constant mean curvature H. Consider $a \in \mathbb{R}^3$. Then*

$$H \int_{\partial M} \langle \alpha(s) \times \alpha'(s), a \rangle ds + \int_{\partial M} \langle N(s) \times \alpha'(s), a \rangle ds = 0, \tag{5.1}$$

where $\alpha = \alpha(s)$ is a parametrization of $x(\partial M)$, N is the Gauss map of M and $N(s) = N(\alpha(s))$.

Proof We follow [LM95, LM96]. On M define the 1-form

$$\omega_p(u) = H\langle x(p) \times u, a \rangle + \langle N(p) \times u, a \rangle, \quad p \in M,\ u \in T_pM.$$

We show that ω is closed. Fix $p \in M$ and let $\{e_1, e_2\}$ be a geodesic frame around p such that $N = e_1 \times e_2$. Since $\sigma(e_1, e_1) + \sigma(e_2, e_2) = 2H$, we have

$$\begin{aligned} d\omega(e_1, e_2) &= e_1\big(H\langle x \times e_2, a\rangle - \langle e_1, a\rangle\big) - e_2\big(H\langle x \times e_1, a\rangle + \langle e_2, a\rangle\big) \\ &= 2H\langle N, a\rangle + H\sigma(e_1, e_2)\langle x \times N, a\rangle - \langle \nabla^0_{e_1} e_1, a\rangle \\ &\quad - H\sigma(e_1, e_2)\langle x \times N, a\rangle - \langle \nabla^0_{e_2} e_2, a\rangle \\ &= 2H\langle N, a\rangle - \langle \nabla^0_{e_1} e_1 + \nabla^0_{e_2} e_2, a\rangle \\ &= 2H\langle N, a\rangle - \big(\sigma(e_1, e_1) + \sigma(e_2, e_2)\big)\langle N, a\rangle \\ &= 0. \end{aligned}$$

Since ω is closed, Eq. (5.1) follows from Stokes' theorem. □

If we consider the unit conormal vector $\nu(s) = N(s) \times \alpha'(s)$, then (5.1) is

$$H \int_{\partial M} \langle \alpha(s) \times \alpha'(s), a \rangle ds + \int_{\partial M} \langle \nu(s), a \rangle ds = 0. \tag{5.2}$$

Remark 5.1.2 Assume that M is not a compact surface. If $C \subset M$ is a closed curve, then the number

$$H \int_C \langle \alpha(s) \times \alpha'(s), a \rangle ds + \int_C \langle \nu(s), a \rangle ds \tag{5.3}$$

depends only the homology class of C. Indeed, if $C' \subset M$ is another closed curve such that C and C' are homologous and $C \cup C'$ bounds a surface $\tilde{M} \subset M$ with $\partial \tilde{M} = C \cup C'$, the flux formula says

$$H \int_{\partial \tilde{M}} \langle \alpha(s) \times \alpha'(s), a \rangle\, ds + \int_{\partial \tilde{M}} \langle \nu(s), a \rangle ds = 0,$$

showing the dependence of (5.3) only on the homology class of C.

Another method to establish (5.1) is the following. Let Y be a Killing vector field of $\mathbb{R}^3$ and consider ϕ_t a 1-parameter family of rigid motions of $\mathbb{R}^3$ associated to Y. Define the variation of the immersion x given by $x_t = \phi_t \circ x$. As the area functional $A(t)$ that measures the area of x_t is invariant by the isometries ϕ_t, it follows that $A'(0) = 0$ and from Proposition 2.1.3 we have

$$2H \int_M \langle N, Y \rangle \, dM + \int_{\partial M} \langle \nu, Y \rangle \, ds = 0. \tag{5.4}$$

Let $a \in \mathbb{R}^3$ be a non-zero vector and define $Y(p) = a$. Then ϕ_t is the 1-parameter family of translations associated to a. Equation (5.4) represents Noether's theorem, which asserts that the linear momentum of a physical system must be conserved [AM08]. We replace the first integral in (5.4) by an integral along the boundary ∂M as follows. Define on M the 1-form β by

$$\beta_p(u) = \big\langle x(p) \times u, a \big\rangle, \quad u \in T_p M.$$

If $\{e_1, e_2\}$ is a geodesic frame around p, then $d\beta(e_1, e_2) = 2\langle N, a \rangle$. Stokes' theorem gives

$$2 \int_M \langle N, a \rangle \, dM = \int_{\partial M} \big\langle \alpha(s) \times \alpha'(s), a \big\rangle ds. \tag{5.5}$$

Combining this with (5.4), we derive (5.2).

Remark 5.1.3 If the surface is the graph of $u \in C^2(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain, the flux formula (5.1) is a direct consequence of the divergence theorem in Eq. (3.3). Indeed, we know that

$$2Ha(\Omega) = \int_{\partial\Omega} \left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \mathbf{n} \right\rangle ds,$$

where $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. Let β be a length-arc parametrization of $\partial\Omega$ such that $\mathbf{n}(s) \times \beta'(s) = e_3 = (0, 0, 1)$. From the expression of N in (3.1) along $\partial\Omega$, we obtain

$$\left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \mathbf{n} \right\rangle = -\big\langle N(s), \mathbf{n}(s) \big\rangle = -\big\langle N(s) \times \beta'(s), e_3 \big\rangle = -\big\langle N(s) \times \alpha'(s), e_3 \big\rangle,$$

where $\alpha(s) = \beta(s) + \langle \alpha(s), e_3 \rangle e_3$ is a parametrization of ∂M, obtaining (5.1).

Consider in (5.4) the other type of Killing vector fields of $\mathbb{R}^3$ associated to the rotations about an axis $a \in \mathbb{R}^3$. Define $Y(p) = p \times a$.

Corollary 5.1.4 *Let M and M' be two compact surfaces with $\partial M = \partial M'$ and $M \cup M'$ an oriented 2-cycle. Let $x : M \to \mathbb{R}^3$ and $y : M' \to \mathbb{R}^3$ be two immersions such that x has constant mean curvature H. If Y is a Killing vector field of*

$\mathbb{R}^3$, *then*

$$\int_{\partial M} \langle \nu, Y \rangle \, ds = 2H \int_{M'} \langle \eta, Y \rangle \, dM',$$

where η *is the restriction of the orientation of* $M \cup M'$ *to* M' *and* ν *is the inward unit conormal vector of* M *along* ∂M.

Proof Consider an orientation on $M \cup M'$ and let N and η be the induced orientations on M and M' respectively. As Y is a Killing vector field, Stokes' theorem gives

$$\int_M \langle N, Y \rangle \, dM + \int_{M'} \langle \eta, Y \rangle \, dM' = 0.$$

Finally, replace the left-hand side of this equation by a boundary integral using (5.4). □

For the Killing vector field $Y(p) = p \times a$, the flux formula represents the conservation of angular momentum.

We apply Corollary 5.1.4 in the next scenario. Suppose that the boundary is a simple closed curve included in a plane P. Let us take M' to be the domain $\Omega \subset P$ bounded by $x(\partial M)$ and let $Y = a$ be a unit vector orthogonal to P. As $\eta = \pm a$, we obtain

$$\int_{\partial M} \langle \nu, a \rangle \, ds = 2H \int_{\Omega} \langle \eta, a \rangle d\Omega = \pm 2Ha(\Omega), \tag{5.6}$$

which is also deduced from (5.2). The important fact of Corollary 5.1.4 is that we can replace Ω by another surface M', the only condition being that its union with M forms a oriented 2-cycle.

Equation (5.6) (or the relation (5.2)) represents the equilibrium between the tension forces of M acting on its boundary with the external forces acting on the surface. The value $\int_{\partial M} \langle \nu, a \rangle \, ds$ is the flux of the surface in the direction of a and, in our context of cmc surfaces, this number does not depend on M but only on H and the boundary curve of the surface. This is the reason that (5.1) is called the *balancing formula* [Kus88, Proposition I.1.8].

The flux formula allows us to show that some geometric configurations of cmc surfaces cannot exist. This is the case for the surface M that appears in Fig. 5.1. This embedded surface spans a boundary curve Γ contained in a plane P, $M \cap \Omega = \emptyset$ and M lies above P around Γ. Indeed, let $W \subset \mathbb{R}^3$ be the 3-domain bounded by M and Ω. The Gauss map N corresponding to positive mean curvature points inside by comparing M and a horizontal plane at the highest point of M. If a is a unit vector orthogonal to P such that $\langle \nu, a \rangle > 0$, then $\eta = -a$ on Ω and the flux formula (5.6) would say

$$0 < \int_{\partial M} \langle \nu, a \rangle \, ds = 2H \int_{\Omega} \langle \eta, a \rangle \, d\Omega = -2Ha(\Omega) < 0,$$

a contradiction.

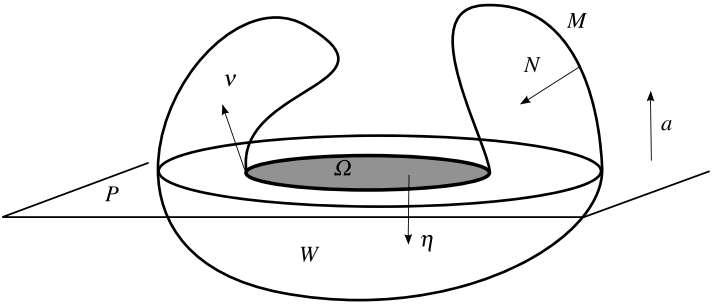

Fig. 5.1 A compact embedded cmc surface that cannot exist by the flux formula

A more general result will prove that if the surface is transverse to P along ∂M, then M lies on one side of P (see Theorem 5.4.1 in Sect. 5.4).

The flux formula adopts in the minimal case the following expression:

Corollary 5.1.5 *Let M be a compact minimal surface immersed in $\mathbb{R}^3$. If $a \in \mathbb{R}^3$,*

$$\int_{\partial M} \langle \nu(s), a \rangle \, ds = 0, \qquad \int_{\partial M} \langle \nu(s), \alpha(s) \times a \rangle \, ds = 0.$$

We then say that the flux vanishes on a minimal surface. This means that there exists an equilibrium of the forces of the surface throughout the connected components of the boundary. In the next result, this equilibrium can be seen when ∂M has two connected components and the surface represents a soap film with free boundary.

Corollary 5.1.6 *Let P_1 and P_2 be two planes of $\mathbb{R}^3$. Let M be a compact minimal surface immersed in $\mathbb{R}^3$ where $\partial M = \Gamma_1 \cup \Gamma_2$ and $\Gamma_i \subset P_i$ are simple closed curves. If the angle between M and P_i is constant along Γ_i, then P_1 and P_2 must be parallel.*

Proof In view of Theorem 3.3.1, we know that the surface is included in the slab (resp. wedge) defined by P_1 and P_2 if the planes are parallel (resp. not parallel). If the angle is 0 or π, M is tangent to P_i and the tangency principle implies that M is a subset of a plane, which is impossible because $P_1 \neq P_2$. This means that the surface is transverse to both planes.

Since the angle between M and P_i is constant, Joachimstal's theorem implies that Γ_i is a line of curvature of M. In particular, $\sigma(\alpha', \nu) = 0$, where α is a parametrization by the arc length of ∂M. Let $b \in \mathbb{R}^3$ be a vector parallel to P_1. Since the vector $\alpha' \times \alpha$ is perpendicular to P_1 along Γ_1, we have

$$\begin{aligned} \int_{\Gamma_1} \langle \nu, b \rangle \, ds &= \int_{\Gamma_1} \langle N \times \alpha', b \rangle \, ds = -\int_{\Gamma_1} \langle N' \times \alpha, b \rangle \, ds \\ &= \int_{\Gamma_1} \sigma(\alpha', \alpha') \langle \alpha' \times \alpha, b \rangle \, ds + \int_{\Gamma_1} \sigma(\alpha', \nu) \langle \nu \times \alpha, b \rangle \, ds = 0. \end{aligned}$$

Hence, if a_i is a unit vector orthogonal to P_i, we find that

$$\int_{\Gamma_i} \nu \, ds = \left(\int_{\Gamma_i} \langle \nu, a_i \rangle \, ds \right) a_i, \quad i = 1, 2.$$

Furthermore, $\langle \nu, a_i \rangle$ is a non-zero constant function because, up to a sign, it represents the cosine of the contact angle and thus $\int_{\Gamma_i} \langle \nu, a_i \rangle \, ds \neq 0$.

Finally, we prove the result. We proceed by contradiction assuming that P_1 and P_2 are not parallel, that is, a_1 is not proportional to a_2. By Corollary 5.1.5, we have the vectorial identity

$$0 = \int_{\Gamma_1 \cup \Gamma_2} \nu \, ds = \left(\int_{\Gamma_1} \langle \nu, a_1 \rangle \, ds \right) a_1 + \left(\int_{\Gamma_2} \langle \nu, a_2 \rangle \, ds \right) a_2.$$

As $\{a_1, a_2\}$ are linearly independent vectors, the integrands must vanish. This contradiction gives the corollary. □

The flux formula allows us to answer to the question posed in the introduction to this chapter concerning the possible values H of an H-surface spanning a given curve. For this purpose, we need the following definition. Let $\alpha = \alpha(s)$ be a closed parametrized curve of period $T > 0$, where s denotes the arclength. Given $a \in \mathbb{R}^3$, $|a| = 1$, define the *algebraic area* of α with respect to a by

$$A(\alpha, a) = \frac{1}{2} \int_0^T \big\langle \alpha(s) \times \alpha'(s), a \big\rangle ds.$$

This number coincides with the algebraic area of the orthogonal projection of α onto a plane P perpendicular to the vector a. If $\Gamma \subset \mathbb{R}^3$ is a closed curve, we represent by $A(\Gamma, a)$ the algebraic area of any arc length parametrization of Γ. Up to a sign, this number is independent of the parametrization of Γ. In the particular case that Γ is a simple closed curve included in a plane P and choosing a as a unit orthogonal vector to P, the number $|A(\Gamma, a)|$ is the area of the planar domain bounded by Γ in P.

Corollary 5.1.7 *Let $\Gamma \subset \mathbb{R}^3$ be a closed curve and let $a \in \mathbb{R}^3$, $|a| = 1$, such that $A(\Gamma, a) \neq 0$. If M is a compact H-surface immersed in $\mathbb{R}^3$ spanning Γ, then*

$$|H| \leq \frac{L}{2|A(\Gamma, a)|}, \tag{5.7}$$

where L is the length of Γ. In particular, if Γ is a simple plane closed curve enclosing a domain Ω of area $a(\Omega)$, then

$$|H| \leq \frac{L}{2a(\Omega)}. \tag{5.8}$$

Inequality (5.7) generalizes the necessary condition obtained by E. Heinz for an immersed H-disk [Hei69].

Proof The flux formula (5.2) is

$$\int_{\partial M} \langle v, a\rangle \, ds = -2HA(\Gamma, a). \tag{5.9}$$

By definition of an immersion with boundary, we have used that the restriction $x_{|\partial M}$ is one-to-one (see Sect. 2.1). It follows that

$$2|H|\big|A(\Gamma, a)\big| \leq \int_{\partial M} ds = L,$$

obtaining (5.7). If Γ is a simple closed curve included in a plane P, we see that

$$|H| \leq \frac{L}{2\bar{A}},$$

where

$$\bar{A} = \sup\big\{\big|A(\Gamma, a)\big| : A(\Gamma, a) \neq 0, |a| = 1\big\}.$$

It is straightforward that the number $\bar{A}$ is attained when a is a unit vector orthogonal to P and, in such case, $|A(\Gamma, a)| = a(\Omega)$. □

Notice that the estimate (5.8) is different from the one obtained in Theorem 3.3.4 because in this theorem the numerator in the estimate is $L(\partial\Omega)$, the length of the boundary of Ω. Even in the case that Γ is a graph on the boundary of a bounded domain Ω, we have $L(\partial\Omega) \leq L$ and equality holds only if $\Gamma = \partial\Omega$.

In the particular case that Γ is a circle, inequality (5.8) gives:

Corollary 5.1.8 *Let M be an H-surface spanning a circle of radius r. Then*

$$0 \leq |H| \leq \frac{1}{r}.$$

The case that the boundary is a circle is special because the examples given by a planar disk and a spherical cap show that H is attained for *all* values of the interval $[-1/r, 1/r]$. Consider the general case. Given a simple closed curve $\Gamma \subset \mathbb{R}^3$, we define

$$J(\Gamma) = \{H \in \mathbb{R} : \text{there exists an } H\text{-surface spanning } \Gamma\}.$$

We are interested in the nature of $J(\Gamma)$ as a subset of $\mathbb{R}$. If $H \in J(\Gamma)$ and M is an H-surface spanning Γ, reversing the orientation on M we obtain that $-H \in J(\Gamma)$. Thus the set $J(\Gamma)$ is symmetric about $0 \in \mathbb{R}$.

We study the particular case that Γ is a simple plane curve. Here the length L of Γ coincides with the length of the boundary of the domain $\Omega \subset \mathbb{R}^2$ bounded

by Γ. Then $J(\Gamma)$ is included in $[-L/(2a(\Omega)), L/(2a(\Omega))]$. There is no reason to expect that all numbers of $[-L/(2a(\Omega)), L/(2a(\Omega))]$ are possible values of mean curvatures of H-surfaces spanning Γ. However we can say something about $J(\Gamma)$. Since Ω is a minimal surface, $0 \in J(\Gamma)$. In Chap. 8 we will see that there exists an $\epsilon > 0$ such that for $H \in (-\epsilon, \epsilon)$, there exists an H-graph on Ω with boundary Γ (Corollary 8.1.16). This means that $(-\epsilon, \epsilon) \subset J(\Gamma)$. In fact, the set of numbers H of H-graphs spanning Γ is an interval $(-H_{\max}, H_{\max})$ (Corollary 8.1.16). However, we do not know how large the set $J(\Gamma) \setminus (-H_{\max}, H_{\max})$ is.

We finish this section using the flux formula to obtain results on uniqueness. For compact surfaces with boundary, the simplest result asserts the uniqueness of graphs for the same mean curvature and the same boundary (Theorem 3.2.6). The case of equality in (5.8) is solved when the surface is a topological disk [Lop99b].

Proposition 5.1.9 *Let $\Gamma \subset \mathbb{R}^3$ be a simple plane closed curve of length L and $a(\Omega)$ denote the area of the planar domain Ω bounded by Γ. Let M be a topological disk and $x : M \to \mathbb{R}^3$ be an immersion with constant mean curvature H and boundary Γ. If $|H| = L/(2a(\Omega))$, then Γ is a circle and the immersion describes a hemisphere.*

Proof Let P be the plane containing Γ and let a be a unit vector orthogonal to P. By (5.2) and since $|A(\Gamma, a)| = a(\Omega)$, we deduce that the inward unit conormal vector is $\nu = \pm a$ and M and P are orthogonal along Γ. By Joachimstal's theorem, ∂M is a line of curvature of M.

There is no loss of generality in assuming that M is the unit complex disk $D = \{z = u + iv \in \mathbb{C} : |z| \leq 1\}$. Consider the Hopf differential $\Phi(dz)^2$ where z are conformal coordinates of M and $\Phi = e - g - 2if$, where e, f and g represent the coefficients of σ. Introducing polar coordinates $z = re^{i\theta}$, we find that

$$\sigma(x_r, x_\theta) = -\frac{r}{2}\sin(2\theta)(e - g) + r\cos(2\theta) f.$$

The imaginary part of Φ is

$$\mathrm{Im}(\Phi) = r^2(e - g)\sin(2\theta) - 2r^2 f\cos(2\theta) = -2r\sigma(x_r, x_\theta).$$

As ∂M is a line of curvature, $\sigma(x_r, x_\theta) = 0$ along ∂D, in particular, $\mathrm{Im}(\Phi) = 0$ along ∂D. Since Φ is holomorphic, $\mathrm{Im}(\Phi)$ is a harmonic function and the maximum principle leads to $\mathrm{Im}(\Phi) = 0$ on D. Thus $\Phi = 0$ and the immersion is umbilical. Consequently, M is a spherical cap intersecting P orthogonally. This shows that $x(M)$ describes a hemisphere and Γ is a circle. □

In this proposition, the assumption on the value of the mean curvature is equivalent to saying that the surface is orthogonal to the boundary plane. When Γ is a circle and the contact angle is constant, we have:

Theorem 5.1.10 *Let M be a compact cmc surface immersed in $\mathbb{R}^3$ spanning a circle Γ contained in a plane P. If the angle between M and P along Γ is constant, then M is a planar disk or a spherical cap.*

Proof Assume without loss of generality that Γ is a circle of radius 1 centered at the origin and contained in the plane $z = 0$. Let $\alpha = \alpha(s)$ be an arc length parametrization of ∂M such that $\{\alpha(s), \alpha'(s), a\}$ and $\{\alpha'(s), \nu(s), N(s)\}$ are positively oriented orthonormal bases. As $\langle N(s), a\rangle$ is constant along ∂M and Γ is a plane curve, we have $\sigma(\alpha'(s), \nu(s)) = 0$. Since $\alpha''(s) = -\alpha(s)$, we obtain

$$\sigma\big(\alpha'(s), \alpha'(s)\big) = -\big\langle N'(s), \alpha'(s)\big\rangle = \big\langle N(s), \alpha''(s)\big\rangle = -\big\langle N(s), \alpha(s)\big\rangle.$$

Using that the function $\langle \nu(s), a\rangle$ is constant, the flux formula (5.2) yields $\langle \nu(s), a\rangle = -H$. By the choice of the two bases, $\langle \nu(s), a\rangle = \langle N(s), \alpha(s)\rangle$ and we deduce that $\sigma(\alpha'(s), \alpha'(s)) = H$. Combining this with $\sigma(\alpha'(s), \nu(s)) = 0$, we conclude that ∂M is a set of umbilical points, in particular, they are not isolated. In view of Theorem 2.2.2, the surface is umbilical and this proves the result. □

This theorem may be compared with Theorem 4.1.16. In both results, the assumption is the capillarity condition along the boundary. However, in Theorem 4.1.16 the boundary is not prescribed but we must assume embeddedness of the surface.

We turn to the case where the boundary is a circle. By Corollary 5.1.8 we know that $|H| \leq 1/r$. F. Brito and R. Earp solved the particular case $|H| = 1/r$, which is now a consequence of the preceding theorem [BE91].

Corollary 5.1.11 *Let Γ be a circle of radius $r > 0$. Then the only compact H-surface immersed in $\mathbb{R}^3$ with $|H| = 1/r$ spanning Γ is the hemisphere of radius r.*

Proof The flux formula (5.2) gives $|\langle \nu, a\rangle| = 1/r$. In particular, the surface is perpendicular with the boundary plane and we apply Theorem 5.1.10. □

5.2 Uniqueness on Right Cylinders

The second setting where we employ the flux formula is motivated by the uniqueness of cmc graphs. Identify $\mathbb{R}^2$ with the Euclidean plane P with equation $z = 0$ and let $e_3 = (0, 0, 1)$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Given a real number H, there exists at most one H-graph on Ω spanning $\partial\Omega$. This result generalizes to the family of embedded surfaces included in the solid right cylinder $\Omega \times \mathbb{R}$ (see Corollary 4.2.2). In this section we consider immersed cmc surfaces contained in $\Omega \times \mathbb{R}$ and with planar boundary [LM95]. The idea behind the proof is to sandwich the surface between two graphs with the same mean curvature and the same boundary. This allows us to compare the slope of the surface with that of the graphs along the boundary and, finally, we apply the flux formula (see [Ser69a] in a similar context for the Plateau problem).

Theorem 5.2.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and assume there exists an H-graph G on Ω with boundary $\Gamma = \partial\Omega$. Denote by G^* the reflection of G about P.*

Consider a compact surface M and let $x : M \to \mathbb{R}^3$ be an immersion with constant mean curvature H spanning Γ. If $x(M) \subset \overline{\Omega} \times \mathbb{R}$, then $x(M)$ is G or G^*.

Proof Note that G is the graph of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, i.e. we permit G to be vertical at some boundary points. We identify M with $x(M)$. If $H = 0$, Theorem 3.3.1 yields $G = M = \overline{\Omega}$. Suppose now that $H \neq 0$. There is no loss of generality in assuming that $H > 0$ and that G lies above the plane P. This means by using Corollary 3.1.7 that the orientation N_G on G points downwards. The graph G^* is an H-surface spanning Γ with the upwards orientation.

Move G by vertical translations $\phi_t(p) = p + te_3$ and let $G_t = \phi_t(G)$. Since M and G are compact sets, $G_t \cap M = \emptyset$ for sufficiently large t. Next let $t \searrow 0$ until G_t first contacts M at $t = t_1$, $t_1 \geq 0$. Let $p \in G_{t_1} \cap M$ be a common point. If $t_1 > 0$, then $p \in int(M)$. If $p \in \partial G_{t_1}$, then $p \in \partial\Omega \times \mathbb{R}$. This means that M is vertical at p, which implies that G_{t_1} is also vertical at p, in particular, G_{t_1} and M are tangent at p and one surface lies on one side of the other. Therefore, independently of whether p is a boundary or interior point of G_{t_1}, the surfaces G_{t_1} and M are tangent at p. We orient M by N so that at p, $N(p)$ agrees with $N_{G_{t_1}}(p)$, where $N_{G_{t_1}}$ is the induced orientation on G_{t_1} by ϕ_{t_1} and G. As $M \geq G_{t_1}$ around p, the tangency principle implies that the mean curvature of M is H and M agrees with G_{t_1} in an open set around p. Extending the set where M and G_{t_1} coincide by using the tangency principle, we obtain $M = G_{t_1}$, which is impossible because $\partial M \neq \partial G_{t_1}$.

In conclusion, $t_1 = 0$ and in the process for moving down G_t, we return to the original position of G and, in addition, G lies above M. If at this position, there is an interior or a boundary tangent point between M and G, then G lies on one side of M. The same argument using the tangency principle tells us that $M = G$ and this proves the result. If, on the contrary, M lies strictly below G along Γ, with the aid of G^*, a similar reasoning shows that $M = G^*$ or M lies strictly above G^* along Γ.

Summarizing the above arguments, we have proved that either $M = G$, or $M = G^*$ or $int(M)$ lies in the 3-domain bounded by G and G^* and M is neither tangent to G nor G^* at some boundary point. Now we prove that the last situation leads to a contradiction. See Fig. 5.2.

Let ν_G, ν denote the inner unit conormal vectors of G and M along Γ, respectively. Since $H > 0$ and G lies above P, the flux formula (5.2) gives in each surface:

$$2H\, a(\Omega) = \int_{\partial\Omega} \langle \nu_G, e_3 \rangle \, ds = \left| \int_{\partial\Omega} \langle \nu, e_3 \rangle \, ds \right|. \tag{5.10}$$

The property that M lies strictly between G and G^* means

$$\langle \nu_{G^*}, e_3 \rangle < \langle \nu, e_3 \rangle < \langle \nu_G, e_3 \rangle$$

along $\partial\Omega$, where ν_{G^*} is the corresponding inward conormal vector of G^* along $\partial G^* = \partial\Omega$. Since G^* is the reflection of G about P, $\langle \nu_{G^*}, e_3 \rangle = -\langle \nu_G, e_3 \rangle$. Then along ∂M we have the strict inequality

$$\left| \langle \nu, e_3 \rangle \right| < \langle \nu_G, e_3 \rangle,$$

obtaining a contradiction with (5.10). □

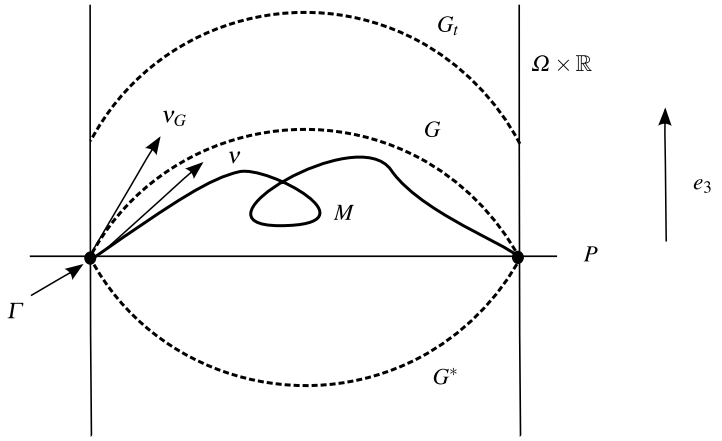

Fig. 5.2 The surface M lies sandwiched between G and G^*

From this theorem, we ask if an immersed cmc surface M included in the cylinder $\Omega \times \mathbb{R}$ and spanning $\partial\Omega$ is actually a graph on Ω. The existence of an H-graph on Ω is assured if H is sufficiently small. Indeed, we shall prove in Chap. 8 that there is a constant $H_0 = H_0(\Omega) > 0$, depending only on Ω, such that for $H \in (-H_0, H_0)$, there exists an H-graph on Ω spanning $\partial\Omega$. Moreover, if Ω is a strictly convex domain, it is possible to estimate this value H_0. More precisely, let κ denote the curvature of $\partial\Omega$ computed with the inward unit normal and let $\kappa_0 = \min\kappa > 0$. Then we will see that $H_0 \geq \kappa_0$. Combining this and Theorem 5.2.1, we have:

Corollary 5.2.2 *Given a planar bounded domain Ω, there is a positive number $H_0 = H_0(\Omega)$ depending only on Ω such that if $|H| \leq H_0$, there exists a unique compact H-surface included in $\overline{\Omega} \times \mathbb{R}$ and with boundary $\partial\Omega$ which is, indeed, a graph on Ω. Furthermore, if $\partial\Omega$ is strictly convex, the result holds provided $\kappa_0 \geq |H|$.*

We remark that the condition that the surface is included in the cylinder $\overline{\Omega} \times \mathbb{R}$ is necessary even for small values of H. For example, given a circle Γ, there are large spherical caps spanning Γ with arbitrary small values of H.

If $H = 0$, Theorem 5.2.1 is trivial because necessarily $x(M) = \overline{\Omega}$. In the minimal case, we consider the case where Γ is not planar. As a consequence of Theorems 3.2.6 and 5.2.1, we have:

Corollary 5.2.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $\Gamma \subset \partial\Omega \times \mathbb{R}$ be a closed curve that is a graph on $\partial\Omega$. If M_1 and M_2 are two minimal surfaces with $\partial M_1 = \partial M_2 = \Gamma$ and $int(M_i) \subset \Omega \times \mathbb{R}$, then $M_1 = M_2$.*

It is known that there exist minimal graphs defined on a convex domain for arbitrary boundary data ([Brn10, Fin65]; Theorem 8.2.3). Using Theorem 3.3.1, we conclude (see also [Ser69a]):

Corollary 5.2.4 *Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain and let $\Gamma \subset \mathbb{R}^3$ be a simple closed curve that is a graph on $\partial\Omega$. Then there is a unique compact minimal surface spanning Γ, which is necessarily a graph.*

5.3 Small Surfaces with Constant Mean Curvature

We turn now to spherical caps. Given a circle Γ of radius $r > 0$ and $|H| \leq 1/r$, the two spherical caps of mean curvature H spanning Γ are included in the closure of a round ball of radius $1/|H|$. The next result of J.L. Barbosa characterizes this property [Bar90].

Theorem 5.3.1 *Let Γ be a circle. Assume $x : M \to \mathbb{R}^3$ is an immersion of a compact surface M spanning Γ and with constant mean curvature $H \neq 0$. If $x(M)$ is included in the closure of a ball of radius $1/|H|$, then $x(M)$ is a spherical cap.*

Proof Choose the orientation N on M so its mean curvature H is positive. Without loss of generality, assume that Γ is included in the plane P with equation $z = 0$. Denote by B the open ball of radius $1/H$ such that $x(M) \subset \overline{B}$. The boundary $S = \partial B$ is a sphere of radius $1/H$ whose mean curvature is H with the inward orientation N_S. First, assume that S and $x(M)$ have a common tangent (interior or boundary) point p. Since $H > 0$, the comparison principle implies that N coincides with N_S at p. The tangency principle implies that both surfaces agree on an open neighborhood of p, and by Theorem 2.2.2, $x(M)$ describes a spherical cap because its boundary is a circle.

Suppose now that $x(int(M)) \subset B$. Moreover, if $x(M)$ intersects S at a point, necessarily, a boundary point, both surfaces are not tangent. Fix $v \in \mathbb{R}^3$. Move S in the direction of v until S touches $x(M)$. If this occurs at a tangent point, the same reasoning as above shows that $x(M)$ is a spherical cap. If not, S can be moved until it touches M at a point of the circle Γ and at this point, M and S are not tangent.

As a conclusion, if $x(M)$ is not a spherical cap, for each direction $v \in \mathbb{R}^3$, we can find a position B_v of the ball B, which is a translation of B in the v-direction, such that the sphere ∂B_v touches M only at a boundary point and both surfaces are not tangent. Thus $x(M) \subset \bigcap\{B_v : v \in \mathbb{R}^3\}$ and all balls B_v have radii $1/H$. By the geometry of the sphere and since Γ is a circle, the surface $x(M)$ lies sandwiched between two small H-spherical caps, or two H-hemispheres, S^+ and S^-, where S^- is the reflection of S^+ through P, $\partial S^+ = \partial S^- = \Gamma$ and S^+ and S^- are not tangent to M along its common boundary: see Fig. 5.3. In particular, $x(M) \neq S^+, S^-$, and $x(int(M)) \subset \Omega \times \mathbb{R}$, where $\Omega \subset P$ is the domain bounded by Γ. Now Theorem 5.2.1 yields a contradiction. □

Related to this theorem, it is the parametric Plateau problem (1.2), which we recall here. Given $H \in \mathbb{R}$, S. Hildebrandt proved that if Γ is a simple closed curve included in a ball B of radius $r > 0$ with $|H| \leq 1/r$, there exists an H-disk spanning Γ and contained in the closure of B [Hil70]. This leads us to make the following definition:

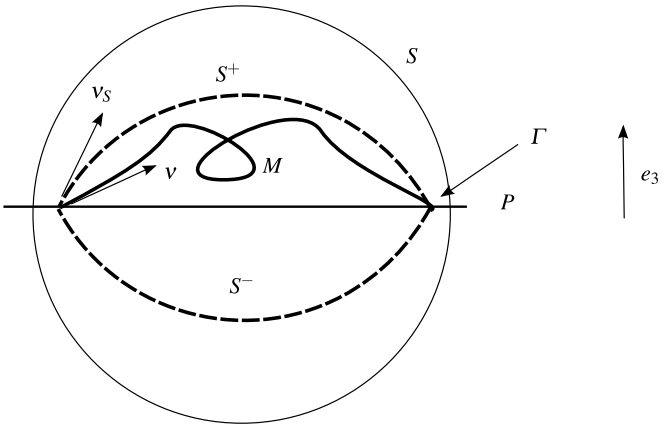

Fig. 5.3 The proof of Theorem 5.3.1. The surface lies between the two small spherical caps S^+ and S^- with the same boundary

Definition 5.3.2 Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M and with constant mean curvature $H \neq 0$. We say that the surface is H-small if $x(M)$ is included in a closed ball of radius $1/|H|$.

In particular, Corollary 3.3.7 asserts that a sphere of radius $r > 0$ is the only closed $1/r$-small surface. Furthermore, Hildebrandt solutions are H-small. It is worth noting that for each $H \in \mathbb{R}$, the spherical H-caps are H-small surfaces and that Theorem 5.3.1 characterizes them among all H-small surfaces spanning a circle. Consider now H-small surfaces bounded by convex plane curves.

Theorem 5.3.3 *Let $\Omega \subset \mathbb{R}^2$ be a (not round) strictly convex domain and let $\kappa_0 > 0$ be the minimum of the curvature of $\partial\Omega$. Then for each real number H such that $|H| \leq \kappa_0$, the only H-small surface in $\mathbb{R}^3$ with boundary $\partial\Omega$ is a graph.*

Proof Denote by P the plane $z = 0$. Theorem 8.3.1 guarantees the existence of an H-graph G on Ω spanning $\partial\Omega$. Let M be an H-small surface with boundary Γ. Consider the ball B of radius $1/|H|$ whose closure contains M and let $S = \partial B$. Proceeding as in the proof of Theorem 5.3.1, we move S in any direction until S intersects M. We assert that S and M do not have a common tangent point. In such a case, the tangency principle would say that an open set of M is part of a sphere and Theorem 2.2.2 would conclude that M is a spherical cap because ∂M is planar. This is a contradiction because Ω is not a round disk.

We denote by $B(p, r)$ the Euclidean ball of radius r and center p and $S(p, r) = \partial B(p, r)$. Move the initial sphere S vertically until its center lies in the plane P, indeed, $B(q, 1/|H|)$ and $q \in P$. We claim that M remains inside $\overline{B(q, 1/|H|)}$. This is because by the tangency principle there cannot exist a tangent point between M and S. On the other hand, the initial sphere S intersects the plane P since $\Gamma \subset \overline{B}$.

Thus when moving S vertically in order that its center lies in P, the radii of the circles obtained by the intersection of P with each vertical displacement of S increase. Since $\overline{\Omega} \subset B \cap P$, it follows that $\overline{\Omega} \subset \overline{B(q, 1/|H|)} \cap P$ without contact between Γ and $S(q, 1/|H|)$.

Now we use a rolling property of planar curves due to Blaschke [Bla56, Kou72].Since the radius of the circle $S(q, 1/|H|) \cap P$ is $1/|H|$ and $1/|H| \geq 1/\kappa_0$, we can move $S(q, 1/|H|)$ in any direction parallel to P touching each point of Γ and $\Omega \subset B(q, 1/|H|) \cap P$. Since $S(q, 1/|H|)$ is vertical at the points of $S(q, 1/|H|) \cap P$, M is contained in $\overline{\Omega} \times \mathbb{R}$. Now Theorem 5.2.1 finishes the proof. □

We call the reader's attention to the fact that the assumption $|H| \leq \kappa_0$ has only been used for the existence of an H-graph on Ω. Hence we can replace this assumption by another one that permits us to establish the existence of such H-graph. In this sense, we refer to Chap. 8 again for other existence results. As a consequence of the proof of Theorem 5.3.3, we have:

Corollary 5.3.4 *Let $H \neq 0$ and let $x : M \to \mathbb{R}^3$ be an H-small surface. Assume that $x(M)$ is not part of a sphere. Then*

$$x(M) \subset \bigcap \left\{ \overline{B(p,r)} : x(\partial M) \subset \overline{B(p,r)}, r \leq \frac{1}{|H|} \right\}.$$

Proof If B is a ball of radius $1/|H|$ containing M, moving B in any direction, the tangency principle guarantees that the sphere ∂B only touches M at boundary points. Thus,

$$x(M) \subset \bigcap \left\{ \overline{B(p, 1/|H|)} : x(\partial M) \subset \overline{B(p, 1/|H|)} \right\}.$$

Consider $B(p, r)$ with $r \leq 1/|H|$ and $x(\partial M) \subset \overline{B(p, r)}$. Increase the radius r until it reaches the value $1/|H|$. From the previous argument and after a translation, $x(M) \subset \overline{B(p, 1/|H|)}$. Decreasing the radius again, and as the mean curvature of the boundary sphere increases with respect to the inward orientation, there are no tangent points with M before we reach the boundary. Then we return to the original ball $B(p, r)$ and $M \subset \overline{B(p, r)}$. □

Theorem 5.3.1 may be extended to the case when the surface is included in a solid right cylinder [Bar91].

Theorem 5.3.5 *Let Γ be a circle. Assume $x : M \to \mathbb{R}^3$ is an immersion of a compact surface M spanning Γ and with constant mean curvature $H \neq 0$. If $x(M)$ is included in the closure of a round cylinder of radius $1/|H|$, then $x(M)$ is a spherical cap.*

Proof Without loss of generality, assume that $H > 0$ and that the radius of Γ is 1, which implies that $H \leq 1$. Denote by C the solid cylinder containing $x(M)$ and

by L the axis of ∂C. Consider a closed hemisphere S^+ of radius $1/H$ such that $\partial S^+ \subset \partial C$. The hemisphere S^+ divides the cylinder C into two domains one of which, denoted C^+, is convex. Similarly, define S^- to be the reflection of S^+ by a plane orthogonal to L and let C^- be the convex connected component of $C \setminus S^-$. By translations, we move S^+ parallel to L in such way that its boundary ∂S^+ remains in ∂C and $x(M)$ is included in C^+. Let us move S^+ towards $x(M)$ in the direction of L until the first time it touches $x(M)$. With the hemisphere S^- we proceed in the same way, moving first along L with $x(M) \cap S^- = \emptyset$ and $x(M) \subset C^-$. Next, displace S^- towards $x(M)$ until it first touches $x(M)$.

If for S^+ or for S^- there is a tangent (interior or boundary) point and as both hemispheres S^+ and S^- have constant mean curvature H with the unit normal points towards C^+ and C^-, respectively, the tangency principle asserts that $x(M)$ is included in a such hemisphere, showing the result because ∂M is a circle.

Assuming the contrary, the hemispheres S^+ and S^- intersect Γ and M lies in $C^+ \cap C^-$. In such case, we claim that S^+ and S^- do intersect. Once the claim is proved, it will follow that the surface $x(M)$ lies in a closed ball of radius $1/H$ and Theorem 5.3.1 concludes the proof.

The proof of the claim is a consequence of the geometry of a circle and a sphere. In the moment of intersection with $x(M)$, let $\tilde{S}^+$ and $\tilde{S}^-$ denote the round spheres containing S^+ and S^-, respectively. Both spheres are included in the solid cylinder C. Consider the plane P containing Γ. Then P cuts S^+ and S^- (resp. $\tilde{S}^+$ and $\tilde{S}^-$) into two arcs Γ^+ and Γ^-, respectively (resp. into two circles $\tilde{\Gamma}^+$, $\tilde{\Gamma}^-$, centered at O^+ and O^-), and Γ is internally tangent to both arcs at points $p^+ \in \Gamma \cap \Gamma^+$ and $p^- \in \Gamma \cap \Gamma^-$.

If P is parallel to L, then $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^-$ have identical radii. It is straightforward that Γ^+ intersects Γ^- and so the same occurs with S^+ and S^-.

Assume now the plane P is not parallel to L. If S^+ and S^- do not intersect, then $\Gamma^+ \cap \Gamma^- = \emptyset$ and there exists a plane Π orthogonal to L separating S^+ from S^-, which leaves on each side of Π the centers O^+ and O^-. As Γ is internally tangent to $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$, the center of Γ lies in the segments joining p^+ with O^+ and p^- with O^-. However this is impossible because Π separates O^+ and O^-. □

Observe that Theorems 5.3.1 and 5.3.5 are the boundary versions of Corollary 3.3.7 and Theorem 3.3.6 respectively.

As in Chap. 3, the next natural step consists in assuming that the surface is included in a slabwhose width is related with the mean curvature H. We know that a spherical cap with mean curvature $H > 0$ lies in a slab of width $2/H$ and that the small spherical cap is included in a slab of width $1/H$. If the surface is embedded, we have [Lop97c]:

Theorem 5.3.6 *Let $H \neq 0$. Let M be a compact embedded H-surface in $\mathbb{R}^3$ spanning a circle. If M lies in a slab of width $1/|H|$, then the surface is a spherical cap.*

Proof After a change of orientation on M, consider $H > 0$. Assume that the slab S is horizontal with $\partial S = P_1 \cup P_2$ two parallel planes. Arguing as in the proof of

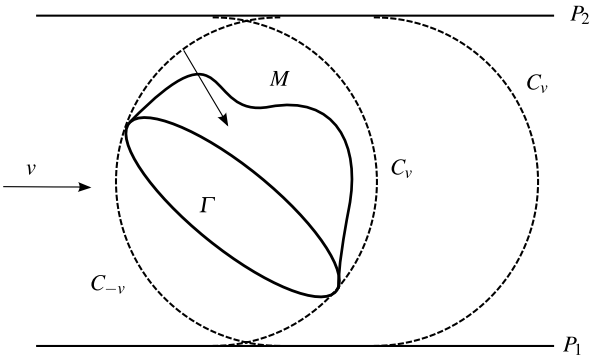

Fig. 5.4 The half cylinders C_v and C_{-v} only touch M at boundary points

Theorem 3.3.9, let v be a horizontal vector and let C_v denote a half-cylinder of radius $1/(2H)$ such that $\partial C_v \subset P_1 \cup P_2$. We adopt the notation used there. Move C_v in the v-direction so that $M \subset C_v^+$ and next, move C_v back towards M. We make use of the same symbol to indicate C_v and each one of its displacements. Taking into account that the mean curvature of C_v with the orientation pointing towards C_v^+ is H, the tangency principle assures us that when C_v first touches M, this must occur at a boundary point and both surfaces are not tangent. See Fig. 5.4. Let P be the plane containing Γ and let $\Omega \subset P$ be the domain bounded by Γ. There are two possibilities depending on the relative position of P and P_i:

1. The plane P is perpendicular to P_i. The preceding argument proves that the boundaries of C_v and C_{-v} reach the plane P and $C_v \cap C_{-v} \neq \emptyset$. In particular, the surface M lies in a solid cylinder of radius $1/(2H)$ and Theorem 5.3.5 implies that the surface is a spherical cap.
2. The plane P is not orthogonal to P_i. In such case, we repeat the argument with all horizontal vectors, concluding

$$M \subset \bigcap \{C_v^+ : v \text{ is parallel to } P_i\}. \tag{5.11}$$

Since Γ is a circle, for each point $p \in \Gamma$ there exists a horizontal vector v such that $\Gamma \subset C_v^+$ and $p \in C_v$. In particular, the open half straight line $L_p \subset P$ from p orthogonal to Γ with $L_p \cap \Omega = \emptyset$ satisfies $L_p \subset \mathbb{R}^3 \setminus C_v^+$. Observe that $\cup_{p\in\Gamma} L_p = P \setminus \overline{\Omega}$. If we take all horizontal vectors and from (5.11), we obtain that $M \cap (P \setminus \overline{\Omega}) = \emptyset$. Then Corollary 4.1.4 proves the result. □

The above property of a circle holds for a non-strictly convex curve Γ, that is, if $\Omega \subset P$ is the domain bounded by Γ, then

$$P \setminus \overline{\Omega} = \cup\{L_p : p \in \Gamma\},$$

where $L_p \subset P$ is the half-line starting at $p \in \Gamma$, orthogonal to Γ and $L_p \cap \Omega = \emptyset$. As a consequence of the proof, we have:

Corollary 5.3.7 *Let $H \neq 0$. Let Γ be a closed convex curve included in a plane P. Let M be a compact embedded H-surface in $\mathbb{R}^3$ with $\partial M = \Gamma$. If M lies in a slab of width $1/|H|$ that is not orthogonal to P, then M lies on one side of P.*

5.4 The Brito, Earp, Meeks and Rosenberg Theorem

In Chap. 4 we have studied embedded cmc surfaces by means of the Alexandrov reflection method. Now we see how combining this technique with the flux formula allows us to obtain a partial answer to Conjecture 2 given in Chap. 1. In view of Theorem 4.1.1, we need to know conditions which ensure that the surface lies on one side of the boundary plane. The next result is due to F. Brito, R. Earp, W. Meeks III and H. Rosenberg [BEMR91]:

Theorem 5.4.1 *Let Γ be a convex closed curve included in a plane P. Let M be a compact embedded cmc surface in $\mathbb{R}^3$ spanning Γ. If M is transverse to P along Γ, then M lies on one side of P.*

Proof There is no loss of generality in assuming that P is the plane $z = 0$ and that a neighborhood of Γ in M lies above the plane P. Let $e_3 = (0, 0, 1)$. Then $\langle \nu, e_3 \rangle > 0$ along ∂M, where ν is the inward unit conormal vector of M along ∂M.

Let us assume for contradiction that M has points on both sides of P. First, we construct a closed embedded topological surface S as follows. Denote by $\Omega \subset P$ the domain bounded by Γ. If $M \cap \Omega = \emptyset$, it suffices to consider $S = M \cup \Omega$. Otherwise, i.e. if $M \cap \Omega \neq \emptyset$, we assume, by transversality, that $M \cap \Omega = D_1 \cup \cdots \cup D_n$, where D_i are topological disks included in Ω: if not, we intersect M with a horizontal plane close to P in such a way that M is transverse to Ω and M continues to be transverse to P along Γ. Each domain D_i defines a small annulus $A_i \subset M$, with $\partial D_i = A_i \cap P$ and $\partial A_i = C_i^+(\epsilon_i) \cup C_i^-(\epsilon_i)$, $\epsilon_i < \epsilon$, where $C_i^+(\epsilon_i)$ and $C_i^-(\epsilon_i)$ are closed curves included in horizontal planes $P_i^+ : z = \epsilon_i$, $P_i^- : z = -\epsilon_i$, respectively. Let $D_i^+(\epsilon_i) \subset P_i^+$ and $D_i^-(\epsilon_i) \subset P_i^-$ be the disks bounded by $C_i^+(\epsilon_i)$ and $C_i^-(\epsilon_i)$, respectively. To define the surface S, we remove from M the annuli A_i and attach to M the horizontal planar domains $D_i^+(\epsilon_i)$ and $D_i^-(\epsilon_i)$ and finally we attach the domain Ω. In case that for some i, j, $\overline{D_i} \subset D_j$, we take distinct values of ϵ_i ($\epsilon_i < \epsilon$) to guarantee that the resulting surface, possible disconnected, is embedded. Let S be the connected component that contains Ω and let $\widetilde{M} = M \cap S$. We point out that the boundary of $\widetilde{M}$ is $\partial\Omega$ and possible curves $C_i^+(\epsilon_i)$ and $C_j^-(\epsilon_j)$.

Let W be the domain bounded by S. We orient S by the unit normal vector field pointing towards W and denote by N the induced orientation on $\widetilde{M}$. The proof is given in a sequence of steps.

Claim 1. There is at most one connected component of $\widetilde{M} \cap (P \setminus \overline{\Omega})$ homologous to Γ in $P \setminus \overline{\Omega}$.

We proceed by contradiction, assuming that C_1 and C_2 are two such curves. Consider a horizontal direction, say, $v = (0, 1, 0)$. Let Π_t denote the 1-parameter

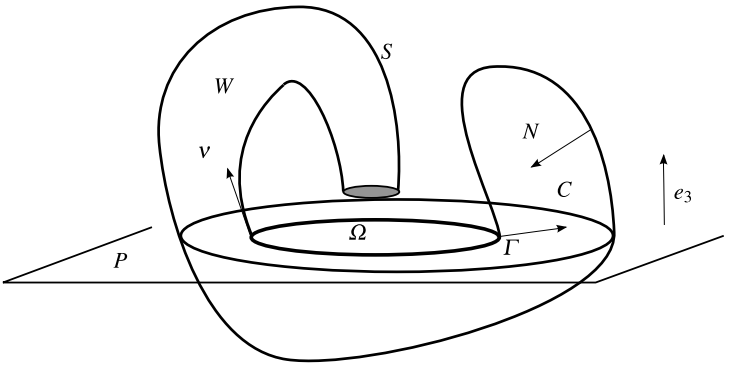

Fig. 5.5 The proof of $\widetilde{M} \cap (P \setminus \overline{\Omega}) = \emptyset$

family of vertical parallel planes with equation $y = t$. After translations, assume that Π_0 intersects Γ and that $\Gamma \cap \Pi_t = \emptyset$ for all $t > 0$. Let $R \subset P$ be a half-line starting from a point of Γ parallel to v and $R \cap \Omega = \emptyset$. Take two points $p_i \in R \cap C_i$, $i = 1, 2$, with $p_i \in \Pi_{a_i}$ and $0 < a_1 < a_2$. Let $\Pi_{\bar{t}}$ be the plane with $\bar{t} = (a_1 + a_2)/2$.

We begin with the Alexandrov method approaching from $t = +\infty$ until the first time $t_0 > 0$ such that Π_{t_0} intersects $\widetilde{M}$. Next we displace the planes Π_t on the left, i.e. $t \searrow 0$, and we reflect the right part of $\widetilde{M}$. With the usual notation employed in Theorem 4.1.1, let t_1 be the first time that $\widetilde{M}^*_{t_1} \not\subset \overline{W}$. By the existence of the points p_1 and p_2, this must happen for a value $t_1 \geq \bar{t}$. When this occurs, there exists a $p \in \widetilde{M}^+_{t_1}$ such that $p^* \in \widetilde{M}^*_{t_1} \cap \widetilde{M}^-_{t_1}$ and both surfaces $\widetilde{M}^*_{t_1}$ and $\widetilde{M}^-_{t_1}$ touch. If $p = p^*$, then T_pM is a horizontal plane. Observe that the point p^* defines a horizontal half-line $(p^*, \infty) = \{p^* + (t - t_1)v : t > t_1\}$. We rule out all the possibilities:

1. $p^* \in \Gamma$. Then $(p^*, \infty) \subset P$. Moreover, by transversality, the tangent plane T_pM is not horizontal, proving that $p \neq p^*$. By the Alexandrov method, we know $(p^*, p) \subset W$ and $(p, \infty) \subset \mathbb{R}^3 \setminus \overline{W}$. Thus (p^*, ∞) intersects M in at most one point. This contradicts that (p^*, ∞) intersects C_1 and C_2.
2. $p^* \in C_i^{\pm}(\epsilon_i)$. Taking ϵ sufficiently small, the half-line (p^*, ∞) must intersect M at least at two points because of the existence of C_1 and C_2. This is a contradiction as above.
3. $\widetilde{M}^*_{t_1}$ and $\widetilde{M}^-_{t_1}$ are tangent at p^*. The tangency principle shows that Π_{t_1} is a plane of symmetry of S. This however contradicts that Γ lies on the left of Π_{t_1}.

Claim 2. $\widetilde{M} \cap (P \setminus \overline{\Omega})$ cannot be formed by a unique curve C homotopic to Γ in $P \setminus \overline{\Omega}$.

If such a curve exists, we arrive at a contradiction using the flux formula as follows. By the comparison principle at the highest point of $\widetilde{M}$ and with the choice of orientation N, the mean curvature H is positive. Since the annulus determined by Γ and C lies in W, the projection of N on P along ∂M points outward Ω as depicted in Fig. 5.5. Let us take a parametrization α of ∂M so that the inner unit conormal vector ν satisfies $\nu = N \times \alpha'$. Since $\langle \nu, e_3 \rangle > 0$, the parametrization of α

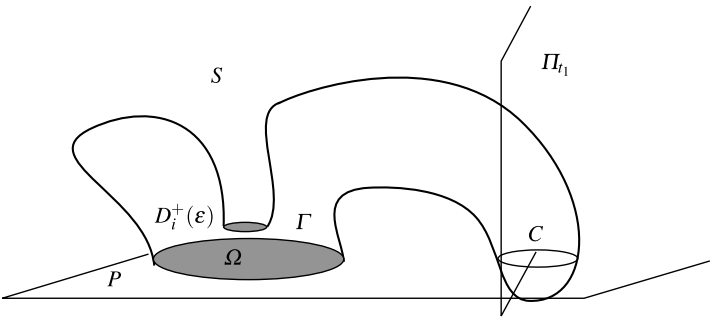

Fig. 5.6 There do not exist curves $C \subset \widetilde{M} \cap P$ null homologous on $P \setminus \overline{\Omega}$

is given by the clockwise orientation of $\partial\Omega$. Thus $\langle \alpha \times \alpha', e_3 \rangle < 0$ and formula (5.1) yields a contradiction.

Observe that in the proof of claims 1 and 2, we have not used that Γ is a convex curve.

Claim 3. The intersection $\widetilde{M} \cap (P \setminus \overline{\Omega})$ has no curves C null homologous on $P \setminus \overline{\Omega}$: see Fig. 5.6.

Assuming the contrary, let R be a half-line starting orthogonally from a point of Γ and intersecting C in at least two points p_1 and p_2. By the convexity of Γ, $R \subset P \setminus \overline{\Omega}$. Let Π be a plane orthogonal to R and equidistant from p_1 and p_2. Without loss of generality, assume that Π is the plane with equation $y = 0$ and that Ω is included in the half space $\{(x, y, z) \in \mathbb{R}^3 : y < 0\}$. We consider the family of planes Π_t with equation $y = t$ and proceed with the reflection process approaching from $t = +\infty$. With the usual notation, let t_1 be the first value where $\tilde{M}^*_{t_1}$ touches $\tilde{M}^-_{t_1}$. The existence of p_1 and p_2 implies $t_1 \geq 0$. If $p^* \in \tilde{M}^*_{t_1} \cap \tilde{M}^-_{t_1}$ is a tangent point, then Π_{t_1} is a plane of symmetry of $\tilde{M}$, contradicting the fact that the boundary of $\tilde{M}$ lies on one side of Π_{t_1}.

The other possibility is that $p^* \in \partial\tilde{M}$. Then the transversality of $\tilde{M}$ together with the fact that W lies above P close to Ω implies that $p^* \in C_i^+(\epsilon_i)$ for some i. We assert that we may choose ϵ sufficiently small such that this case, namely, $p^* \in C_i^+(\epsilon_i)$ for some i, cannot occur. On the contrary, for $\epsilon_n \to 0$ we find a sequence $\{q_n\} \subset \tilde{M}^+_{(t_1)_n}$ and $\{q_n^*\} \subset \partial\tilde{M} \setminus \Gamma$, where q_n^* is the reflected point of q_n through $\Pi_{(t_1)_n}$. Letting $n \to \infty$, and passing to a subsequence if necessary, $\{q_n\} \to q$ and $\{q_n^*\} \to q^*$. As $\epsilon_n \to 0$, $q \in P$ and $q^* \in \bigcup_{i=1}^n \partial D_i$. As all points q_n lie in $\tilde{M}^+_{(t_1)_n}$, the limit point q lies in some plane Π_t, with $t \geq 0$. Since $q^* \in \Omega$, the reflection process implies that the open segment (q^*, q) is included in W, a contradiction because necessarily (q^*, q) must meet Γ.

In conclusion, we have proved that $\widetilde{M} \cap (P \setminus \overline{\Omega}) = \emptyset$. There are two possibilities.

1. If $M \cap \Omega = \emptyset$, then $\widetilde{M} \cap (P \setminus \overline{\Omega}) = \emptyset$, that is, $\widetilde{M} = M$ and we have proved $M \cap (P \setminus \overline{\Omega}) = \emptyset$, showing the result.
2. If $M \cap \Omega \neq \emptyset$, we extend $\widetilde{M}$ up to Ω by adding the annulus bounded by $\partial D_i \cup C_i^+(\epsilon_i)$ and again denote by $\widetilde{M}$ the new surface. Since $\widetilde{M} \cap (P \setminus \overline{\Omega}) = \emptyset$, the flux

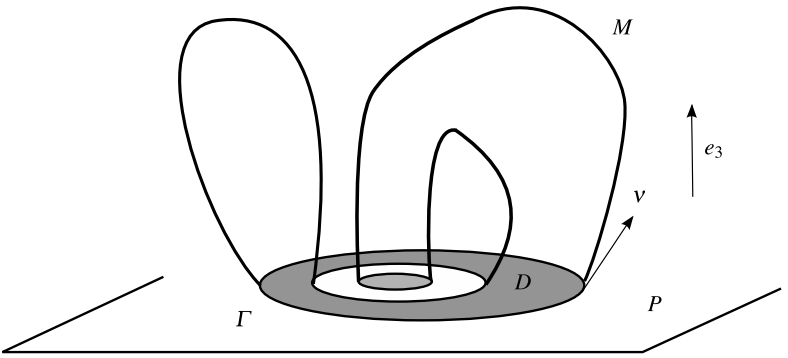

Fig. 5.7 The final contradiction in Theorem 5.4.1

formula (5.1) implies

$$2H\big(a(\Omega) - a(D)\big) = \int_{\partial \widetilde{M}} \langle \nu, e_3 \rangle \, ds, \tag{5.12}$$

where $D \neq \emptyset$ is an open set of $\cup_i D_i$: see Fig. 5.7. Since $\partial M \subsetneq \partial \widetilde{M}$, we have

$$\int_{\partial \widetilde{M}} \langle \nu, e_3 \rangle \, ds > \int_{\partial M} \langle \nu, e_3 \rangle \, ds. \tag{5.13}$$

But the flux formula applied to the initial surface M gives

$$\int_{\partial M} \langle \nu, e_3 \rangle \, ds = 2Ha(\Omega),$$

yielding a contradiction with (5.12) and (5.13). □

Corollary 5.4.2 *Let M be a compact embedded cmc surface spanning a circle Γ. If M is transverse to the plane containing the circle along Γ, then M is a spherical cap.*

In connection with Theorem 5.4.1, the authors formulate in [BEMR91] the following conjecture:

Conjecture. A compact embedded cmc surface of $\mathbb{R}^3$ spanning a convex plane curve and on one side of the boundary plane is a topological disk.

A partial answer to this conjecture was given in Theorem 4.2.1. See also Corollary 6.4.5 for embedded surfaces with large volume.

We point out that the assumption on the convexity of the curve has been used in order to prove that there are no curves in $\widetilde{M} \cap P$ null homologous on $P \setminus \overline{\Omega}$. If Γ is not convex, we have the following result [BEMR91].

Theorem 5.4.3 *Let Γ be a simple closed curve included in a plane P. Let M be a compact embedded cmc surface in $\mathbb{R}^3$ spanning Γ. Suppose that M is transverse to P along Γ and M lies above P around Γ. Let M^+ and M^- be the parts of M above and below P, respectively. Then*:

1. *$int(M) \cap \Omega = \emptyset$.*
2. *M^+ is connected and no connected component of ∂M^+ lies completely outside of the convex hull of Γ.*
3. *M^- is a graph on the domain bounded by $\partial M^+ \setminus \Gamma$ in P.*

As a consequence:

Corollary 5.4.4 *Let M be a compact embedded cmc surface whose boundary is a simple closed curve Γ contained in a plane P. If the angle between M and P is constant along Γ, then M is a planar domain or a spherical cap.*

Proof If $H = 0$, we know that M is a planar domain. If $H \neq 0$, the angle is not 0 or π by the flux formula (5.2) since $\langle \nu, e_3 \rangle$ is constant along ∂M. In particular, the surface is transverse to P along $\partial \Gamma$. By Theorem 5.4.3, $M \cap \Omega = \emptyset$. Thus $M \cup \Omega$ defines a 3-domain $W \subset \mathbb{R}^3$ where $\Omega \subset W$.

There is no loss of generality in assuming that M lies above P around Γ. As M^- is a graph on a domain of $P \setminus \overline{\Omega}$, the domain W lies above P in a neighborhood of Ω. We employ the Alexandrov method as in Theorem 4.1.16 and the proof follows the same steps. We only remark that the reflection of a point $p \in M^+_{t_1} \setminus \Gamma$ cannot touch Γ because the segment (p^*, p) would be included in W, a contradiction since $\Omega \subset W$. □

Chapter 6
The Area and the Volume of a Constant Mean Curvature Surface

In this chapter we address the study of the area and the volume of a compact cmc surface with boundary. We shall see how the control of the area or the volume provides information concerning the geometry of the surface. In order to motivate the type of problems that we consider here, let $\Omega \subset \mathbb{R}^2$ be a bounded domain. The domain Ω is itself a minimal surface with boundary $\Gamma = \partial\Omega$ and its volume is $V = 0$. For small values of H close to 0, there exist H-graphs on Ω spanning Γ and whose volume is also small. We may expect that given Ω and a sufficiently small volume V, the only compact cmc surface with boundary $\partial\Omega$ and volume V is a graph. This evidence is supported when we blow air across a planar wire. Once a plane closed wire is introduced into a soapy solution, the first surface that appears is precisely the planar domain bounded by the wire. If we blow a small amount of air against the surface, the bubble formed looks like a graph attached to the wire.

We shall derive a formula giving a lower bound of the area in terms of the height of the surface with respect to the plane containing the boundary. We remark that the estimates of the height appearing in Theorem 2.3.5 for cmc graphs and Corollary 4.2.5 for embedded surfaces similarly relate the height to the value of the mean curvature. We will show that the solution of the isoperimetric problem for a convex planar domain is a graph if the volume is sufficiently small.

In the case where the volume is large, we expect that the surface looks like a large spherical cap. We will prove this fact in Sect. 6.4 for the particular case where the surface is embedded spanning a convex curve and the surface lies on one side of the boundary plane.

Some of the results in this chapter and their proofs are taken from [LM96, RR96].

6.1 A Monotonicity Formula for the Area

Consider a plane $P \subset \mathbb{R}^3$ given by $P = \{p \in \mathbb{R}^3 : \langle p - p_0, a\rangle = 0\}$ with $|a| = 1$. If $S \subset \mathbb{R}^3$, define the part of S above (resp. below) the plane P as the set $S^+ = \{p \in S : \langle p - p_0, a\rangle > 0\}$ (resp. $S^- = \{p \in S : \langle p - p_0, a\rangle < 0\}$). Analogously, the height

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_6,
© Springer-Verlag Berlin Heidelberg 2013

h^+ (resp. h^-) of S above (resp. below) P is $h^+ = \sup\{\langle p - p_0, a\rangle : p \in S^+\}$, (resp. $h^- = -\inf\{\langle p - p_0, a\rangle : p \in S^-\}$).

Theorem 6.1.1 *Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M with constant mean curvature H and whose boundary is included in a plane P. Then*

$$h^+ \leq \frac{A^+|H|}{2\pi}, \tag{6.1}$$

where A^+ is the area of $x(M)^+$. Equality holds if and only if $x(M)^+$ is a planar domain of P or a spherical cap.

Proof If $H = 0$, the result is trivial by Theorem 3.1.12. Assume now $H \neq 0$ and let us orient M with the unit normal vector field N so that $H > 0$. After a change of coordinates, assume that $P = \{p \in \mathbb{R}^3 : \langle p, a\rangle = 0\}$, where $|a| = 1$.

Suppose that $x(M)^+ \neq \emptyset$, on the contrary (6.1) is trivial. Consider the height function f with respect to P defined by $f(p) = \langle x(p), a\rangle$. Then $f = 0$ on ∂M. On the other hand, if $p \in M$ and $v \in T_pM$ is a tangent vector at p,

$$\big\langle (\nabla f)_p, v\big\rangle = \big\langle (dx)_p(v), a\big\rangle. \tag{6.2}$$

Thus $p \in M$ is a critical point of f if and only if the vector a is perpendicular to M at p, that is, if $N(p) = \pm a$. Then the set C of critical points of f is included in $\mathcal{N}_b \cap \mathcal{N}_c$, where

$$\mathcal{N}_b = \big\{p \in M : \big\langle N(p), b\big\rangle = 0\big\}, \qquad \mathcal{N}_c = \big\{p \in M : \big\langle N(p), c\big\rangle = 0\big\},$$

and $\{a, b, c\}$ is an orthonormal basis of $\mathbb{R}^3$. In fact, $\mathcal{N}_b$ and $\mathcal{N}_c$ are the nodal lines of the functions $\langle N, b\rangle$ and $\langle N, c\rangle$. Appealing to (2.24), both functions $\langle N, b\rangle$ and $\langle N, c\rangle$ satisfy the Jacobi equation. Following [Che76], each nodal line of the operator $\Delta + |\sigma|^2$ is, by the immersion, the image of a circle or a segment of $\mathbb{R}$ and thus, C has zero measure in M. In this case, the coarea formula gives

$$A'(t) = -\int_{\Gamma(t)} \frac{1}{|\nabla f|} ds_t, \quad t \geq 0,$$

where $A(t)$ is the area of $M(t) = \{p \in M : f(p) \geq t\}$ and ds_t is the arclength in $\Gamma(t) = \partial M(t) = \{p \in M : f(p) = t\}$ [Fed69]. Here, for almost t, $M(t)$ is a surface. If $L(t)$ denotes the length of the curve $\Gamma(t)$, the Cauchy-Schwarz inequality gives

$$L(t)^2 \leq \int_{\Gamma(t)} \frac{1}{|\nabla f|} ds_t \int_{\Gamma(t)} |\nabla f| ds_t = -A'(t) \int_{\Gamma(t)} |\nabla f| ds_t. \tag{6.3}$$

But (6.2) yields

$$|\nabla f|^2 = 1 - \langle N, a\rangle^2 = \big\langle \nu^t, a\big\rangle^2$$

along $\Gamma(t)$, where ν^t is the inner conormal vector of $M(t)$ along $\Gamma(t)$. Since $M(t)$ lies above the plane $P_t = \{p \in \mathbb{R}^3 : \langle p, a\rangle = t\}$, we have $\langle \nu^t, a\rangle \geq 0$. Hence

$$|\nabla f|_{|\Gamma(t)} = \langle \nu^t, a\rangle.$$

Then (6.3) can be expressed as

$$L(t)^2 \leq -A'(t) \int_{\Gamma(t)} \langle \nu^t, a\rangle \, ds_t, \quad t \in \mathbb{R}. \tag{6.4}$$

If $t \geq 0$, then $x(\Gamma(t)) \subset P_t$ and the flux formula (5.2) gives

$$\int_{\Gamma(t)} \langle \nu^t, a\rangle \, ds_t = 2H \big|A\big(\Gamma(t), a\big)\big|.$$

Substituting this into (6.4), we obtain

$$L(t)^2 \leq -2HA'(t)\big|A\big(\Gamma(t), a\big)\big| \tag{6.5}$$

for almost $t \geq 0$. Let $\Omega_1(t), \ldots, \Omega_{n_t}(t)$ be the bounded domains that determine the closed curve $x(\Gamma(t))$ in the plane P_t and let $A_i(t)$ be the area of the domains $\Omega_i(t)$, $1 \leq i \leq n_t$. Then

$$A\big(\Gamma(t), a\big) = \epsilon_1 A_1(t) + \cdots + \epsilon_{n_t} A_{n_t}(t),$$

where $\epsilon_i \in \mathbb{Z}$ are the indices of the curves $\partial\Omega_i(t)$ with respect to an interior point and a given orientation. Therefore

$$\big|A(\Gamma(t), a\big| \leq |\epsilon_1| A_1(t) + \cdots + |\epsilon_{n_t}| A_{n_t}(t).$$

On the other hand, if $L_i(t)$ is the length of the boundary of $\Omega_i(t)$, then $L(t) = |\epsilon_1| L_1(t) + \cdots + |\epsilon_{n_t}| L_{n_t}(t)$ and hence

$$L(t)^2 \geq \epsilon_1^2 L_1(t)^2 + \cdots + \epsilon_{n_t}^2 L_{n_t}(t)^2.$$

Combining these two inequalities with (6.4), for almost $t \geq 0$, we have

$$\epsilon_1^2 L_1(t)^2 + \cdots + \epsilon_{n_t}^2 L_{n_t}(t)^2 \leq -2HA'(t)\big(|\epsilon_1| A_1(t) + \cdots + |\epsilon_{n_t}| A_{n_t}(t)\big).$$

The classical isoperimetric inequality in the domain $\Omega_i(t)$ gives $L_i(t)^2 \geq 4\pi A_i(t)$. Since $\epsilon_i^2 \geq |\epsilon_i|$, we find that

$$2\pi \leq -HA'(t).$$

Integrating this inequality from 0 until $h^+ = \max_{p \in M} f(p) \geq 0$, we conclude that

$$2\pi h^+ \leq H\big(A(0) - A\big(h^+\big)\big) = HA^+,$$

obtaining the desired inequality.

If equality holds in (6.1), then $|\nabla f|_{|\Gamma(t)} = \langle \nu^t, a\rangle$ is a constant for all $t \geq 0$, $n_t = 1$, $|\epsilon_i| = 1$ and the equality in the isoperimetric inequality implies that $x(\Gamma(t))$ is a circle. Therefore, in a neighborhood of the point of M where the maximum height is attained, the immersion is an embedding whose image is a cmc surface bounded by a circle and above the boundary plane. Corollary 4.1.3 establishes that this part of the surface is a spherical cap and, in particular, the umbilical points of the surface M are not isolated. Appealing to Theorem 2.2.2, the immersion is umbilical, which implies that it is a spherical cap because $x(M)^+ \neq \emptyset$ and the boundary is a plane curve. □

The area rate of the surface in relation to the height t given by the inequality

$$A'(t) \leq \frac{-2\pi}{H}$$

justifies the name *monotonicity formula* for the area. For cmc graphs, the estimate was proven in [PSS77] and [Spe81, Chap. 11]. See also [ST79].

A first consequence of the estimate (6.1) applies to a closed cmc surface. In this case, we take any plane to estimate the height of the surface to conclude:

Corollary 6.1.2 *The diameter $\delta(M)$ of a closed H-surface M in $\mathbb{R}^3$ satisfies*

$$\delta(M) \leq \frac{A|H|}{2\pi},$$

where A is the area of M. Moreover, equality holds if and only if M is a round sphere.

With the same assumption as Theorem 6.1.1, we derive

$$h^- \leq \frac{A^-|H|}{2\pi} \tag{6.6}$$

where A^- denotes the area of $x(M)^-$. As a consequence

Corollary 6.1.3 *Let M be a compact H-surface immersed in $\mathbb{R}^3$ whose boundary is contained in a plane P. If A is the area of M, then M lies in a slab parallel to P of width less than or equal to $A|H|/(2\pi)$. Equality holds if and only if M is a spherical cap.*

If the boundary of the surface is not planar, the next result was shown by H.C. Wente for a cmc disk immersed in $\mathbb{R}^3$ [Wen80b]; compare also Corollary 4.2.6.

Corollary 6.1.4 *Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M with constant mean curvature H. Assume that the boundary is included in a ball $B(p, r)$.*

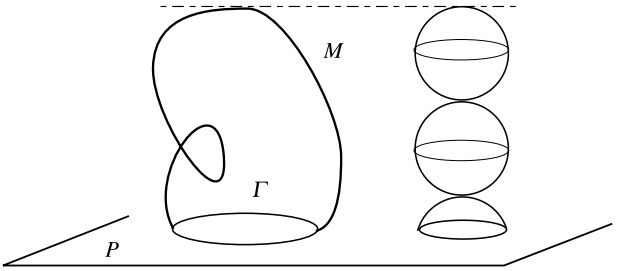

Fig. 6.1 The H-surface M on the *left* has larger area than the set of H-spheres on the *right*

Then

$$x(M) \subset B\left(p, r + \frac{A|H|}{2\pi}\right),$$

where A is the area of the surface.

Proof We use (6.1) and a similar argument to that of Corollary 4.2.6. □

Returning to Theorem 6.1.1, we point out that the right cylinder of radius $1/|H|$ and height h has area $2\pi h/|H|$. Thus Theorem 6.1.1 now formulates as:

Corollary 6.1.5 *Let $H \neq 0$. A compact H-surface immersed in $\mathbb{R}^3$ bounded by a plane curve has an area greater than or equal to the area of a right cylinder of radius $1/|H|$ and with the same height, unless the surface is a spherical cap of radius $1/|H|$, in which case, both areas agree.*

The final part of the above statement is an old result of Archimedes (see, for instance, [Hea63, p. 293]).

On the other hand, if we wish to compare the area of a surface M with respect to the standard surface with the same mean curvature, Theorem 6.1.1 establishes that a compact H-surface with planar boundary has an area greater than the area of a stack of spheres of radius $1/|H|$, whose height is the same than the surface: see Fig. 6.1.

We end this section by comparing the height estimate (6.1) with the one given in Theorem 2.3.5 for cmc graphs. The bound $h \leq 1/|H|$ is obtained now as a consequence of estimate (6.1) if the domain is simply connected. In fact, we have the following improvement.

Corollary 6.1.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain and $H \neq 0$. Let M be an H-graph on Ω with boundary $\partial\Omega$ and denote by L the length of $\partial\Omega$. If $|H| < 2\pi/L$, then the height h of M satisfies*

$$h \leq \frac{1}{|H|}\left(1 - \sqrt{1 - \frac{H^2 L^2}{4\pi^2}}\right).$$

Proof For immersed disks M in $\mathbb{R}^3$, an isoperimetric inequality due to Barbosa and do Carmo gives

$$L^2 - 4\pi A + H^2 A^2 \geq 0, \tag{6.7}$$

where L is the length of ∂M and A is the area of M [BC78]. In this case, M is a disk and (6.7) implies

$$A \leq \frac{2\pi - \sqrt{4\pi^2 - H^2 L^2}}{H^2} \quad \text{or} \quad A \geq \frac{2\pi + \sqrt{4\pi^2 - H^2 L^2}}{H^2}.$$

For $H = 0$, the graph is precisely the domain Ω and the left inequality holds for $H = 0$. By continuity, this inequality is valid for all H-graphs and thus

$$\frac{A|H|}{2\pi} \leq \frac{1}{|H|}\left(1 - \sqrt{1 - \frac{H^2 L^2}{4\pi^2}}\right).$$

This inequality together with (6.1) yields the result. □

6.2 Consequences of the Height Estimate

The height estimate (6.1) allows us to obtain information on the shape of a cmc surface when there is some control concerning its area.

Theorem 6.2.1 *Let $H \neq 0$. Let M be a compact H-surface immersed in $\mathbb{R}^3$ with planar boundary Γ. If its area A satisfies*

$$A \leq \frac{\pi}{H^2},$$

then M lies in the right solid cylinder determined by the convex hull of Γ.

Proof Assume $H > 0$ and that Γ is included in the plane P with equation $z = 0$. By Corollary 6.1.3 and the assumption on A, the surface M lies in a slab S parallel to P of width less than $1/H$. Let $\Gamma^c \subset P$ be the convex hull of Γ, $\Omega^c \subset P$ the domain bounded by Γ^c and let the cylinder be $\Omega^c \times \mathbb{R}$.

Assume, on the contrary, that there exists a $p \in M$ whose orthogonal projection on P, say, p^*, does not belong to $\overline{\Omega^c}$. Assume $q \in \Gamma^c$ is the point minimizing the distance between p^* and Γ^c. Let $R \subset P$ be the half-line starting at q and passing through p^*. We adopt the notation of Theorem 5.3.6. Denote by C_R the half-cylinder of radius $1/(2H)$ whose axis of revolution L lies in P and is orthogonal to R. Position the concave part of C_R so that it points to M. Since the cylinder C_R lies inside S and ∂C_R are two parallel into lines contained in ∂S, C_R separates the slab S into two connected components. Then we assume that $M \subset C_R^+$, the convex component of $S \setminus C_R$, and in addition, we position C_R so that $p \in C_R$. Observe that

$C_R \cap \Gamma = \emptyset$. Let us take on C_R the orientation whose mean curvature is H, and thus, pointing towards C_R^+.

Move C_R in the direction of R so far until it does not touch M. Next we come back until the first intersection point p_1 between C_R and M. Denote by C_R^* the cylinder C_R at this time. This occurs before or at the same time that C_R^* reaches its original position C_R because at the beginning, $p \in C_R \cap M$. If L' is the line $C_R^* \cap P$, then $\overline{\Omega^c}$ lies in the connected component of $P \setminus L'$ where C_R^* does not intersect. In conclusion, the point p_1 must be interior to M. Now we apply the tangency principle to conclude that C_R^* and M intersect in an open set around p_1. Then $M \subset C_R^*$, which is impossible because the boundary of M is a plane curve. This contradiction proves the result. □

The assumption $A \leq \pi/H^2$ in this theorem is not optimal, as is demonstrated by a small spherical cap. More precisely, given a circle Γ of radius 1, the area A_- of the small spherical cap of mean curvature H spanning Γ is

$$A_- = \frac{2\pi}{H^2}\big(1 - \sqrt{1 - H^2}\big).$$

Then

$$A_- H^2 = 2\pi\big(1 - \sqrt{1 - H^2}\big)$$

and the surface is included in the solid cylinder defined by Γ. The inequality $A_- H^2 \leq \pi$ holds only if $|H| \leq \sqrt{3}/2$.

When the boundary is convex, we obtain:

Corollary 6.2.2 *Let Γ be a closed convex curve contained in a plane P. Let M be a compact embedded H-surface in $\mathbb{R}^3$ spanning Γ. If its area A satisfies $AH^2 \leq 2\pi$, then M lies on one side of P and if $AH^2 \leq \pi$, then M is a graph.*

Proof Assume $AH^2 \leq 2\pi$. In view of Corollary 6.1.3, the surface M lies in a slab S parallel to P of width less than $1/|H|$ and we apply Corollary 5.3.7. Suppose now that $AH^2 \leq \pi$. The estimate (6.1) implies that M lies in a slab symmetric with respect to P and of width less than $1/|H|$. By using half-cylinders of radii $1/(2|H|)$ as in Theorem 6.2.1, the interior of M lies in the solid right cylinder orthogonal to P with cross-section Ω. Finally, Theorem 4.2.1 shows that M is a graph on Ω. □

Corollary 6.2.3 *Planar disks and spherical caps are the only compact embedded H-surfaces in $\mathbb{R}^3$ spanning a circle and with area A satisfying*

$$AH^2 \leq 2\pi.$$

If $H \neq 0$, we can rephrase this corollary by saying that the only compact embedded H-surface spanning a circle and with area less or equal than of a hemisphere with the same mean curvature is the spherical cap.

Corollary 6.2.4 *Let Γ be a convex plane curve enclosing a domain Ω. Let M be a compact embedded H-surface in $\mathbb{R}^3$ spanning Γ. If the area A of M satisfies*

$$A \leq \frac{8\pi a(\Omega)^2}{L^2},$$

then M lies on one side of the boundary plane P.

Proof Combining inequality (5.8) and estimate (6.1), we deduce that the surface lies in a slab parallel to P of width less than

$$\frac{A|H|}{2\pi} \leq \frac{4|H|a(\Omega)^2}{L^2} \leq \frac{1}{|H|}.$$

Now we apply Corollary 6.2.1 and finally, Theorem 3.1.11 gives the result. □

6.3 Surfaces with Small Volume

In the preceding section, we have seen that the monotonicity formula (6.1) provides some control on the shape of a compact embedded cmc surface, at least if there are upper estimates of its area. Consider now the algebraic volume V of a cmc surface. The motivation in this section has its origin in the isoperimetric problem. Given a planar domain Ω, we want to investigate in this section those solutions of the isoperimetric problem associated to Ω, namely, surfaces minimizing the area among all surfaces that together with Ω enclose a fix volume.

On the other hand and for small values of H, we know that there are H-graphs on Ω spanning $\partial\Omega$. Furthermore, these graphs enclose a small volume. It is natural to expect that the solutions of the isoperimetric problem are graphs when the prescribed volume is small enough. This section is devoted to the proof of this fact assuming that Ω is convex. First we need some results concerning the volume and the height of a cmc surface.

Proposition 6.3.1 *Let $x : M \to \mathbb{R}^3$ be an immersion of a compact surface M with boundary included in a plane P. If the mean curvature H is constant, then*

$$2HV = \int_M \left|\nabla\langle x, a\rangle\right|^2 dM, \tag{6.8}$$

where V is the volume of the surface and a is a unit vector orthogonal to P.

Proof Let N denote the Gauss map and define on M the 1-forms α and β by

$$\alpha_p(v) = \langle (dx)_p(v), a\rangle\langle x(p), a\rangle,$$
$$\beta_p(v) = \langle (dx)_p(v) \times N(p), x(p) \times a\rangle\langle x(p), a\rangle,$$

for every $p \in M$ and $v \in T_pM$. Compute the codifferentials $\delta\alpha$ and $\delta\beta$. If $\{e_1, e_2\}$ is a geodesic frame around $p \in M$, then

$$\delta\alpha(p) = \sum_{i=1}^{2} \sigma(e_i, e_i)\langle N(p), a\rangle + \sum_{i=1}^{2} \langle a, e_i\rangle^2 = 2H\langle x, a\rangle\langle N, a\rangle + \left|\nabla\langle x, a\rangle\right|^2,$$

$$\begin{aligned}\delta\beta(p) &= \sum_{i=1}^{2} \langle e_i \times N(p), e_i \times a\rangle\langle x(p), a\rangle + \sum_{i=1}^{2} \langle e_i \times N(p), x(p) \times a\rangle\langle e_i, a\rangle \\ &= 3\langle x, a\rangle\langle N, a\rangle - \langle N, x\rangle.\end{aligned}$$

Taking into account that H is constant and since α and β vanish on ∂M, we have

$$\int_M \left|\nabla\langle x, a\rangle\right|^2 dM + 2H \int_M \langle N, a\rangle\langle x, a\rangle \, dM = 0.$$

$$3\int_M \langle N, a\rangle\langle x, a\rangle \, dM - \int_M \langle N, x\rangle \, dM = 0.$$

Combining both equations, we deduce (6.8). □

Corollary 6.3.2 *Let M be a compact H-surface immersed in $\mathbb{R}^3$ with boundary included in a plane P and consider the orientation corresponding to $H \geq 0$. Then the algebraic volume V with this orientation satisfies $V \geq 0$. Moreover, $V = 0$ if and only if the immersion is minimal and its image lies in P.*

Formula (6.8) also gives rise to the following two upper estimates for the height and the area of a cmc graph, which are of interest in themselves.

Theorem 6.3.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let M be an H-graph on Ω with boundary $\partial\Omega$, $H \neq 0$. Assume $a(\Omega) < \pi/H^2$. Then the height h of M satisfies*

$$h \leq \frac{a(\Omega)|H|}{2(\pi - a(\Omega)H^2)}. \tag{6.9}$$

Proof Let M be the graph of a function u. There is no loss of generality in assuming that $u > 0$ on Ω. Let N be the orientation on M pointing downwards, which means that H is positive. Let $e_3 = (0, 0, 1)$. We know $|\nabla\langle x, e_3\rangle|^2 = 1 - \langle N, e_3\rangle^2$ for $x \in M$. Then (6.8) gives

$$2H \int_\Omega u \, dx \, dy = A - \int_M \langle N, e_3\rangle^2 \, dM.$$

Using (2.3), (6.1) and $\langle N, e_3\rangle^2 \leq |\langle N, e_3\rangle|$, we obtain

$$\frac{2\pi h}{H} \le A \le 2H \int_\Omega u\, dxdy + \int_M \langle N, e_3\rangle^2\, dM \le 2hH\, a(\Omega) + \int_M \big|\langle N, e_3\rangle\big|\, dM$$
$$= 2hH\, a(\Omega) + \int_\Omega 1\, dxdy = (2hH + 1)\, a(\Omega),$$

obtaining the desired inequality (6.9). □

The second estimate is due to R. Finn.

Theorem 6.3.4 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let $u \in C^2(\overline{\Omega})$ be a solution of (3.2) and $M = \text{graph}(u)$. Then the area A of M satisfies*

$$A \le \big(3 + 2|H|h\big)a(\Omega) + hL,$$

where

$$h = \max_{\partial\Omega} |u|, \qquad L = length(\partial M).$$

Proof We adopt the notation used in the proof of Proposition 6.3.1. The same computations give (set $N_3 = \langle N, e_3\rangle$):

$$2H \int_M u N_3\, dM + \int_M \big(1 - N_3^2\big)\, dM = -\int_{\partial M} u\langle \nu, e_3\rangle\, ds.$$

The height estimates of Theorem 2.3.5 gives $|u| \le h + 1/|H|$. Thus we derive:

$$\begin{aligned} A &= \int_M N_3^2\, dM - 2H \int_M u N_3 + \int_{\partial M} u\langle \nu, e_3\rangle\, ds \\ &\le a(\Omega) + 2|H| \int_\Omega |u|\, d\Omega + h\left|\int_{\partial M} ds\right| \\ &\le a(\Omega) + 2|H| \int_\Omega \left(h + \frac{1}{|H|}\right) d\Omega + hL \\ &= (3 + 2|H|h)a(\Omega) + hL. \end{aligned}$$

□

We now state and prove the main result in this section.

Theorem 6.3.5 *Let $\Gamma \subset \mathbb{R}^3$ be a convex closed plane curve. There exists a positive number V_Γ depending only on Γ such that for any real number V with $0 \le |V| \le V_\Gamma$, there is a cmc graph with boundary Γ and volume V. In addition, and up to reflections about P, this graph is the only compact cmc surface immersed in $\mathbb{R}^3$ with boundary Γ and volume V.*

The proof is developed in several steps. A first result is the existence of compact H-graphs with small values of H, which was used in the preceding chapter.

Lemma 6.3.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then there exists a constant $H_0 = H_0(\Omega) > 0$ such that if $|H| < H_0$, there exists an H-graph on Ω with boundary $\partial\Omega$.*

Proof See Corollary 8.1.6. □

Lemma 6.3.7 *Let Γ be a closed convex curve included in a plane P and let M be a compact surface. Let $x_k : M \to \mathbb{R}^3$, $k \in \mathbb{N}$, be a sequence of immersions with constant mean curvature H_k with $x_k(\partial M) = \Gamma$. Let A_k and V_k denote the area and the volume of $x_k(M)$, respectively. If $\lim_{k\to\infty} V_k = 0$, then*

$$\lim_{k\to\infty} A_k = a(\Omega) \quad \textit{and} \quad \lim_{k\to\infty} H_k = 0,$$

where $a(\Omega)$ is the area of the planar domain $\Omega \subset P$ bounded by Γ.

Proof In view of (5.7), $\{H_k\}$ is bounded. The assumption concerning the volume implies

$$\lim_{k\to\infty} H_k V_k = 0. \tag{6.10}$$

Now we recall the Minkowski formula for a surface with constant mean curvature. Integrating Eq. (2.23) and using the definition of the volume of an immersion, we obtain

$$A - 3HV = -\frac{1}{2}\int_{\partial M} \langle x, \nu\rangle \, ds, \tag{6.11}$$

where ν is the inner unit conormal vector of M along ∂M. Let $\mathbf{n}$ be the inner unit normal vector to $\partial\Omega$ in the plane P. For each immersion x_k, the identity (6.11) gives

$$A_k - 3H_k V_k = -\frac{1}{2}\int_{\partial M} \langle x_k, \nu_k\rangle \, ds_k = -\frac{1}{2}\int_{\partial M} \langle x_k, \mathbf{n}\rangle\langle \mathbf{n}, \nu_k\rangle \, ds_k, \tag{6.12}$$

where ν_k is the inner unit conormal vector of $x_k(M)$ along Γ. As the curve Γ is convex, and after a suitable choice of $\mathbf{n}$, the support function $\langle x_k, \mathbf{n}\rangle$ is negative on ∂M. Furthermore $\langle \nu_k, \mathbf{n}\rangle \le 1$ and so

$$A_k - 3H_k V_k \le -\frac{1}{2}\int_{\partial M} \langle x_k, \mathbf{n}\rangle \, ds_k = a(\Omega). \tag{6.13}$$

We claim that

$$A_k \ge a(\Omega). \tag{6.14}$$

For this, it suffices to prove that $x_k(M)$ orthogonally projects on a region that includes Ω. Assuming the contrary, if there is a point $p \in \Omega$ such that the straight line passing through p perpendicularly to the plane P does not touch the surface $x_k(M)$, then the curve $x_k(\partial M) = \Gamma$ cannot be homologous to zero in the shadow of $x_k(M)$ on P. However the boundary ∂M is nullhomologous in M and this is a contradiction.

Taking limits in (6.12) and (6.13), we deduce from (6.10) and (6.14) that

$$\lim_{k\to\infty} A_k = a(\Omega),$$

which establishes the first of our results. Furthermore, it follows from (6.12) that

$$a(\Omega) = \lim_{k\to\infty} -\frac{1}{2}\int_{\partial M} \langle x_k, \mathbf{n}\rangle \langle \mathbf{n}, \nu_k\rangle \, ds_k.$$

This fact, together with (6.13), gives

$$\lim_{k\to\infty} \int_{\partial M} -\langle x_k, \mathbf{n}\rangle \big(1 - \langle \mathbf{n}\nu_k\rangle\big) \frac{ds_k}{ds}\, ds = 0,$$

where ds is the arclength element for Γ as a space curve of $\mathbb{R}^3$. Since the boundary curve is convex, the above integrands are not negative and Fatou's lemma implies

$$\liminf_{k\to\infty} \left\{ -\langle x_k, \mathbf{n}\rangle \big(1 - \langle \mathbf{n}, \nu_k\rangle\big) \frac{ds_k}{ds} \right\} = 0.$$

But $\lim_{k\to\infty}\langle x_k, \mathbf{n}\rangle (ds_k/ds) < 0$ because Γ is convex. Then we assume, by passing if necessary to a subsequence, that $\lim_{k\to\infty}\langle \nu_k, \mathbf{n}\rangle = 1$. Let a be a unit vector of $\mathbb{R}^3$ orthogonal to the plane P. Then $\lim_{k\to\infty}\langle \nu_k, a\rangle = 0$. Hence, the flux formula (5.6) and the Lebesgue bounded convergence theorem show that $\lim_{k\to\infty} H_k = 0$. As this holds for every sequence, the second conclusion of lemma follows. □

Proof of Theorem 6.3.5 Consider a sequence of immersions spanning Γ as in the preceding lemma. Let h_k be the height of each surface with respect to the plane P. Inequality (6.1) assures us that $\lim_{k\to\infty} h_k = 0$. Moreover, Lemma 6.3.7 implies $A_k H_k^2 < \pi$ and by Corollary 6.2.2 we deduce that the surface must be a graph.

Take H with $|H| < H_0$, where H_0 is the positive number given in Lemma 6.3.6. We know that there exist H-graphs on Ω spanning $\partial\Omega$ provided $|H| \le H_0$ and by uniqueness, the surface must be such a graph. This completes the proof. □

We recall the definition of an isoperimetric region.

Definition 6.3.8 Let D be a compact surface of $\mathbb{R}^3$ with boundary ∂D. Fix a number $V > 0$. An isoperimetric region of $\mathbb{R}^3$ with respect to (D, V) is an open region $W \subset \mathbb{R}^3$ of volume V with

$$\partial W = D \cup M$$

and M is a surface of least area among all such surfaces. The set M is said to be a solution of the isoperimetric problem with respect to (D, V).

If $D = \emptyset$, we go back the classical isoperimetric problem, whose solution is the sphere [Swz84, Stn42]. Results of F.J. Algrem and J.E. Taylor establish that there always exists a solution of the isoperimetric problem [Alm76, Tay76].

Theorem 6.3.9 *Given a compact surface D in $\mathbb{R}^3$ and a positive number V, there exists a solution of the isoperimetric problem with respect to (D, V). Furthermore, this solution is an embedded surface.*

It is worth noting that this theorem does not provide any information on the topology of the isoperimetric solution. Theorem 2.1.7 asserts that the solution of the isoperimetric problem is stable because it is a minimizer of the area.

As a consequence of Theorem 6.3.5,

Corollary 6.3.10 *Let Ω be a convex planar domain. Then there exists a positive number V_Ω such that for any real number $0 \leq V \leq V_\Omega$, the isoperimetric region corresponding to the pair $(\overline{\Omega}, V)$ is the domain bounded by $G \cup \Omega$, where G is a cmc graph on Ω with $\partial G = \partial\Omega$.*

Remark 6.3.11 By means of a symmetrization argument, M. Koiso proved that given a circle Γ and $V \in \mathbb{R}$, flat disks and spherical caps are the only immersed compact surfaces bounded by Γ enclosing a volume V and with least area among all immersed compact surfaces with the same boundary and volume [Koi95].

6.4 Surfaces with Large Volume

In this section we consider cmc surfaces with sufficiently large volume. We have proved in Sect. 6.3 that a cmc surface spanning a convex curve is a graph if the volume is very small. In a similar setting, if we now increase the volume enough, we expect the surface to resemble a large spherical cap. This question is currently open and in this section we will give a partial answer assuming that the surface is embedded and included in one side of the boundary plane. In addition, we will give a positive answer to the Brito et. al. conjecture under the assumption of large volume.

We introduce the following notation. Let P be the plane with equation $z = 0$, $e_3 = (0, 0, 1)$ and $P^+ = \{p \in \mathbb{R}^3 : \langle p, e_3\rangle \geq 0\}$. Let P_t denote the horizontal plane with equation $z = t$. Denote by $B(p, r)$ the Euclidean ball centered at p of radius $r > 0$ and $D(p, r) = B(p, r) \cap P$ if $p \in P$. If $p = O$ is the origin, we simply write $B(r)$ and $D(r)$, respectively.

Let M be a compact embedded H-surface included in P^+ with $\Gamma = \partial M \subset P$. Let $\Omega \subset P$ be the domain bounded by Γ and let W be the bounded domain of $\mathbb{R}^3$ determined by $M \cup \Omega$. Let $W' = W \cup (\Omega \times (-\infty, 0])$. Consider the orientation on M pointing towards W, which implies, by the comparison principle with planes approaching from above, that $H > 0$.

We make use of an argument similar to that in Theorem 4.2.1. By means of the method of moving planes with P_t approaching from $t = +\infty$, there are two possibilities. If during the entire reflection process, $M_t^* \subset \overline{W'}$ holds for all $t > 0$, then the surface is a graph on Ω. If not, there exists a first time $t_1 > 0$ such that $M_{t_1}^* \setminus \partial M_{t_1}^* \not\subset \overline{W'}$. In such a case, we deduce the following:

1. The first contact occurs between a point $q \in M_{t_1}^+$ and its reflection q^* with respect to P_{t_1}, which is a boundary point $q^* \in \Gamma$. Moreover, the contact between $M_{t_1}^*$ and M at q^* is transverse.
2. The surface M_t^+ is a graph on P_t for all $t \geq t_1$.
3. The highest point of M lies in the cylinder $\Omega \times (0, \infty)$.
4. If h denotes the height of M, then $h < 2/H$ and $h - t_1 \leq 1/H$.

We employ the Alexandrov method by vertical planes as in Theorem 4.1.1. If $v \in \mathbb{R}^3$ is a horizontal unit vector, consider $\Pi_t = \{p \in \mathbb{R}^3 : \langle p, v \rangle = t\}$ and let $\{\Pi_t : t \in \mathbb{R}\}$ be the 1-parameter family of vertical planes orthogonal to v. The reflection method starts with planes approaching from $t = +\infty$. Since the surface lies in P^+, we continue until Π_t touches Γ at $t = t_0$ and $M_{t_0}^* \subset \overline{W}$. Then we can continue reflecting slightly further obtaining $\Gamma_t^* \subset \overline{\Omega}$ for $t \in (t_0 - \epsilon, t_0)$ for some $\epsilon > 0$. Assume that $\partial\Omega$ is a convex curve. If we take all horizontal vectors, the above planes Π_{t_0}, each of which depends on a horizontal vector v, touch all points of $\partial\Omega$ by the convexity of $\partial\Omega$. This proves that $M \setminus (\Omega \times [0, \infty))$ is a graph on the right cylinder $\partial\Omega \times [0, \infty)$ in the sense that any ray emanating outward orthogonally from any point of $\partial\Omega \times \mathbb{R}$ intersects M at most once. In particular, the distance between each connected component of $M \setminus (\Omega \times [0, \infty))$ to $\partial\Omega \times [0, \infty)$ is less than or equal to $1/H$ by Theorem 2.3.5. This proves that M is included in a compact subset of $\mathbb{R}^3$ whose size depends only on Ω and H (this fact was already proven in Corollary 4.2.7).

In the next lemmas we shall use results of curvature estimates and compactness theorems for H-surfaces: we refer to [GT01, Whi87].

Lemma 6.4.1 *Let $H > 0$ and let M be an H-graph on Ω with boundary $\partial\Omega$. Then there exist uniform area and curvature estimates on compact subsets of P^+ depending only on H.*

Proof Up to a reflection about P, assume that M lies over P and since H is positive, N points downwards. Consider $M = \text{graph}(u)$. We know by Theorem 2.3.5 that the function $f = H\langle x, e_3 \rangle + \langle N, e_3 \rangle$ satisfies $\Delta f \geq 0$. Denote by M_t^+ the part of M above the plane P_t. Assume $M_\epsilon^+ \neq \emptyset$. On $M_{\epsilon/4}^+$ we have $f \leq H\epsilon/4$. On the other hand, since $M_{\epsilon/2}^+ \subset M_{\epsilon/4}^+$, $f \leq H\epsilon/4$ in $M_{\epsilon/2}^+$. Using $\Delta f \geq 0$ again, we find that

$$H\langle x, e_3 \rangle + \langle N, e_3 \rangle \leq H\epsilon/4$$

in $M_{\epsilon/2}^+$. Then

$$\langle N, e_3 \rangle \leq H\frac{\epsilon}{4} - Hu \leq H\left(\frac{\epsilon}{4} - \frac{\epsilon}{2}\right) = -\frac{H\epsilon}{4}$$

in $M_{\epsilon/2}^+$. Let $\Omega(\epsilon) = \{x \in \Omega : u(x) \geq \epsilon\}$. Let β be the minimizing curve between $\Omega(\epsilon)$ and $\partial\Omega(\epsilon/2)$ and denote by $L(\beta)$ its length, which we now compute. If $\alpha = \alpha(s)$ is the curve in $M_{\epsilon/2}^+$ whose orthogonal projection is β and if α is parameterized

by the arclength, then

$$L(\beta) = \int_0^L \sqrt{1 - \langle \alpha'(s), e_3 \rangle^2} ds \geq \int_0^L \left| \langle N, e_3 \rangle \right| ds \geq L \frac{H\epsilon}{4},$$

where L is the length of α. But L is greater than the distance between the planes $P_{\epsilon/2}$ and P_ϵ, hence $L(\beta) \geq H\epsilon^2/8$. Let $\delta = H\epsilon^2/16$. Thus, for each $x \in \Omega(\epsilon)$, $D(x, \delta) \subset \Omega(\epsilon/2)$. Since we have control of u and $|\nabla u|$ (or equivalently of $\langle N, e_3 \rangle$) in $\Omega(\epsilon/2)$, the standard theory of elliptic equations guarantees $C^{2,\alpha}$ estimates for u on $\Omega(\epsilon)$ depending on H and ϵ. □

Lemma 6.4.2 *Let $\Gamma \subset P$ be a strictly convex closed curve. Then there exists an $r > 0$ depending only on Γ such that if M is a compact embedded surface spanning Γ and included in P^+, there is a point $p \in \Omega$, depending on M, such that $D(p, r) \subset \Omega$ and $M \cap (D(p, r) \times (0, \infty))$ is a graph over $D(p, r)$.*

Proof If M is a graph on Ω, the result is immediate. Assuming the contrary, let $t_1 > 0$ be the height when $M_{t_1}^*$ first touches Γ at the point $q^* \in \Gamma$. We know that $M_{t_1}^+$ is a graph on P_{t_1} and that $\{q^*\} \times (0, 2t_1) \subset W$. Denote by κ_0 and κ_1 the minimum and the maximum of the curvature κ of Γ. Given a point of Γ, there is a circle C of radius $1/\kappa_0$ tangent to Γ at this point and Γ lies in the closed round disk defined by C. On the other hand, given $0 < \rho < 1/\kappa_1$, Γ has a tubular neighborhood of radius ρ.

Let v be a horizontal vector and consider $\{\Pi_t : t \in \mathbb{R}\}$ the 1-parameter family of vertical planes perpendicular to v foliating $\mathbb{R}^3$. With the usual notation, assume that $\Pi_t \cap \Omega = \emptyset$ for large enough t. Apply the Alexandrov reflection method by planes Π_t approaching from $t = +\infty$. Then we continue until Π_t touches $\partial\Omega$ and we keep reflecting until at least a distance $\rho/2$ beyond the first time the planes meet Γ, in particular, at least a distance $\rho/2$ beyond the first time the vertical plane meets the circle C. If $\vec{n}_{q^*}$ is the inward normal to C at q^*, there is an open set $V \subset P$ of $\vec{n}_{q^*}$ formed by horizontal unit vectors v where we apply Alexandrov's method as above and $M_t^* \subset W$. This open set of horizontal vectors depends only on C (that is, on κ_0 and κ_1). For each v, denote by $J(v)$ the open segment in Ω joining q^* and its reflected point by vertical planes orthogonal to v up to a distance of at least $\rho/2$. The set of these segments only depends on the geometry of Ω. Now, we take $r > 0$ such that a disk $D(p, r)$ is included in $\bigcup_{v \in V} J(v)$. By the Alexandrov reflection method about Π_t, we know that $J(v) \times (0, 2t_1) \subset W$, for each $v \in V$. Thus $D(p, r) \times (0, 2t_1) \subset W$. If we consider $D(p, r) \times (0, \infty)$, we conclude that the intersection with M only occurs with points above the height $t = t_1$, that is, with points of $P_{t_1}^+$. But we know that $M_{t_1}^+$ is a graph on Ω. This ends the proof. □

The next lemma is the key result of this section.

Lemma 6.4.3 *Let M_n be a sequence of compact embedded 1-surfaces included in P^+ with $\partial M_n \subset P$. Assume $\partial M_n \subset D(r_n)$ with $r_n \to 0$. Then there exists a subsequence that either converges to the origin O or to the sphere of radius 1 included in*

P^+ tangent to P at O. In the first case, the convergence of surfaces is on compact subsets, and in the second case, the convergence is smooth on compact subsets of $\mathbb{R}^3 \setminus \{O\}$.

Proof We have proved that M_n is included in a compact subset depending only on Ω and, in this case, $H = 1$. Let $r > 0$. For large n, the Alexandrov method by vertical planes with the surface M_n shows that we can continue reflecting by vertical planes until $\partial D(r)$. Moreover the parts outside $D(r) \times [0, \infty)$ are graphs and, by Lemma 6.4.1, there exist uniform area and curvature estimates. If we take horizontal planes, we reflect until at least $t = 1/H = 1$. A similar argument as in Lemma 6.4.1 shows that there exist area and curvature estimates for $(M_n)^+_{1+\delta}$. These estimates assert that there exists a subsequence of M_n, again named M_n, which converges on compact subsets of $\mathbb{R}^3 \setminus I$, $I = \{0\} \times [0, 1]$. We have two possibilities concerning the limit surface M; either $M = \emptyset$ or M is a surface included in $\mathbb{R}^3 \setminus I$ with constant mean curvature $H = 1$.

1. If $M = \emptyset$ then M_n is uniformly close to I. In particular, M_n is an H-small surface ($H = 1$) and as $\partial M_n \subset B(r_n)$, Corollary 5.3.4 implies that $M_n \subset B(r_n)$. This proves that M_n converges to 0 as $r_n \to 0$.
2. Assume $M \neq \emptyset$. The limit surface is embedded in $\mathbb{R}^3 \setminus I$ because we can apply the Alexandrov method to each surface M_n with vertical planes approaching from infinity at least up to $\partial D(r)$ where r is as close to 0 as we wish. Furthermore, we arrive with horizontal planes P_t until at least a height $1/H = 1$. Fix $r > 0$. Let v be a horizontal vector and Π_t be the vertical planes orthogonal to v where t indicates the signed distance to the z-axis. We know that the reflection method works at least until the planes arrive at $\partial D(r)$. Letting $r \to 0$, the surface M contains the origin O and it is invariant under reflection in Π_0, i.e., M is invariant under reflection in the vertical plane containing the z-axis and perpendicular to v. Since this holds for any horizontal vector, the surface M is rotational and the axis of revolution is the z-axis. Moreover, the height is at most 2. This means that M is a sphere of radius 1 containing the origin O and whose axis of revolution is the z-axis. □

We are now in a position to prove the main result of this section (a generalization for hyperbolic space was established by B. Nelli and B. Semmler in [NeSe99, Sem00]).

Theorem 6.4.4 *Given a strictly convex closed curve Γ in P, there exists an $H(\Gamma) > 0$ such that if $0 < |H| < H(\Gamma)$, any compact embedded H-surface spanning Γ and included in P^+ is a topological disk. Moreover, either M is a graph on Ω or M satisfies the following properties*:

1. *$M \cap (\Omega \times (0, \infty))$ is a graph on Ω.*
2. *$M \setminus (\Omega \times (0, \infty))$ is a graph on the right-cylinder $\Gamma \times (0, \infty)$.*

Proof If M is an H-small surface, Theorem 5.3.3 asserts that M is a graph on Ω if H is sufficiently small. Therefore, we shall assume that M is not an H-small surface.

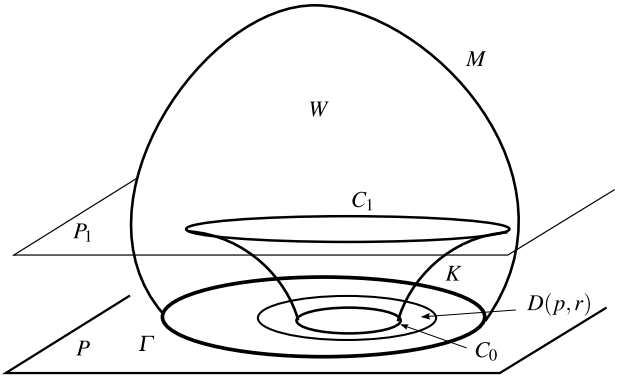

Fig. 6.2 The catenoid K in the proof of Theorem 6.4.4. If the surface M is not an H-small surface, the proof shows that K is included in the domain W for small enough H

Let $p \in \Gamma$ and $r > 0$ be the number given by Lemma 6.4.2. Fix $0 < \rho < \{r, 1/\kappa_0\}$. Let $C_0 = \partial D(p, \rho)$ and consider the catenoid intersecting P along C_0 making a contact angle greater than $\pi/2$ and such that the part of this catenoid over P is a graph on $P \setminus D(p, \rho)$. Let K be the intersection of this catenoid with the slab $\{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq 1\}$ and let $\partial K = C_0 \cup C_1$, where C_1 is a horizontal circle at height $z = 1$. See Fig. 6.2.

Let $V = \{v \in P : C_0 + v \subset \Omega\}$. Let R be sufficiently large so that the disk $D(R) + a$ in the plane P_1 contains $C_1 + v$ for all $v \in P$. Consider a sequence M_n of H_n-surfaces under the hypothesis of the theorem with $H_n \to 0$. We apply Lemma 6.4.3 for the renormalized surfaces $H_n M_n$, which are 1-surfaces with $\Gamma_n = H_n \Gamma \to 0$ and M_n is not an H_n-small surface. Lemma 6.4.3 implies that $H_n M_n$ converges to the sphere $S \subset P^+$ of radius 1 through the origin O. In particular, $(H_n M_n)^+_{1+\delta}$ is close, with respect to the C^k-distance, to the hemisphere $S \cap \{(x, y, z) \in \mathbb{R}^3 : z \geq 1\}$.

We apply Lemma 6.4.3 and the convergence on $\mathbb{R}^3 \setminus \{O\}$. Fix a height equal to $1/2$ over P. For small enough H_n, $M_n \cap (D(R) \times [1/2, \infty))$ is a graph on $D(R)$ with height larger than $1/H_n$. Using again the convergence on $\mathbb{R}^3 \setminus \{O\}$ of Lemma 6.4.3, the reflection of this part of M_n through horizontal planes is a graph uniformly close on $D(R)$ and with height less than $1/2$. Moreover, $M_n \cap (D(R) \times [1/2, 1/H_n]) = \emptyset$. Thus $K + e_3$ and $C_1 + te_3$, $t \in [0, 1]$, are included in W. Since $C_0 + te_3 \subset W$, $0 < t \leq 1$, we show that we can move down the catenoid $K + a$ until its original position and with the property that there do not exist contact points between $K + ta$ and M_n. Indeed, if there is a contact, this must occur at an interior common point. If we orient the catenoid pointing towards W, as for M_n, then the catenoid lies above M_n around the tangent point, which is a contradiction because K is a minimal surface and $H_n > 0$.

Similarly, as $C_1 + v \subset W$, the tangency principle assures us that we can move K horizontally, by $K + v$, $v \in V$, and the catenoid $K + v$ is included in W. By the choice of ρ, for each $q \in \Gamma$, there is a $v \in V$ such that $C_0 + v$ is tangent to Γ at q

and $C_0 + v \subset \overline{\Omega}$. Move K horizontally again by $K + tv$, $0 \leq t \leq 1$. The tangency principle implies that there do not exist contact points between $K + tv$ and M_n and so, $K + v$ touches Γ at the point q. As a conclusion, all horizontal translations $K + v$, $v \in V$, are included in W. As $D(r) \times (0, 1] \subset W$ by Lemma 6.4.2, we see that $\Omega \times (0, 1] \subset W$. On the other hand, we know that $M_n \cap (D(R) \times [1/2, \infty))$ is a graph above $D(R)$ of height greater than $1/H_n$. As $\overline{\Omega} \subset D(R)$, we deduce that $M_n \cap (\Omega \times (0, \infty))$ is a graph on Ω. Furthermore, we have proved that $M_n \setminus (\Omega \times (0, \infty))$ is a graph on $\partial\Omega \times [0, \infty)$. Both facts prove that M is a topological disk. □

Finally, we express this theorem in terms of the volume and the area of the surface and, as announced in the introduction to this section, we shall deduce that if the volume is sufficiently large, the surface must be a topological disk. In fact, by Lemma 6.4.3, it looks like a large spherical cap by the C^∞-convergence.

Corollary 6.4.5 *Let $\Gamma \subset P$ be a strictly convex curve. Then there exist constants $V(\Gamma), A(\Gamma) > 0$ depending only on Γ such that any compact embedded H-surface spanning Γ and included in P^+ is a topological disk whenever its volume is greater than $V(\Gamma)$ or its area is greater than $A(\Gamma)$.*

Proof Let M be a compact embedded H-surface spanning Γ. In view of Corollary 4.2.6 we know that M is included in a ball $B(p, r + 2/|H|)$, where $r > 0$ is a number such that $\Gamma \subset B(p, r)$. Then the volume of M satisfies $0 < V \leq (4\pi/3)(r + 2/|H|)^3$. If V is sufficiently large, for example, $V > V(\Gamma)$, with

$$V(\Gamma) = \frac{8\pi}{3H(\Gamma)} + \frac{4\pi r}{3},$$

then $|H| < H(\Gamma)$, where $H(\Gamma)$ is the value given in Theorem 6.4.4. This implies that M is a topological disk.

For the area, we employ the Minkowski formula (6.11):

$$A = 3HV - \frac{1}{2}\int_{\partial M} \langle x, \nu \rangle \, ds.$$

We know that the boundary part on the right-hand side is bounded in terms of Γ and that H is bounded by (5.8). Hence it follows that if A is large enough, the value of V exceeds $V(\Gamma)$, obtaining the result. □

Corollary 6.4.5 is related to Corollary 6.3.10, where it was proven that given a convex boundary curve and for small volume, the surface is a graph, in particular, a topological disk. In Corollary 6.3.10, there were no assumptions on the embed-

dedness of the surface nor was it assumed that the surface lies on one side of the boundary plane.

Remark 6.4.6 In the Plateau problem for parametric surfaces (1.2), H. Brézis and J.M. Coron studied the behavior of large solutions when H is fixed and the boundaries Γ_n converge to the origin O. They proved that if the area of the solutions M_n remains bounded, a subsequence converges to O or to a finite (connected) union of spheres of radius 1, at least one of them containing O [BC85].

Chapter 7
Constant Mean Curvature Disks with Circular Boundary

We study the case where the boundary of a cmc surface is a circle. In previous chapters, we have obtained some results under the assumption of embeddedness. In this chapter we focus on Conjecture 1 of Chap. 1 and we will assume that the topology of the surface is the simplest one, namely, a disk.

We have seen how the control of the area or the volume of a cmc surface allows us to understand the geometry of the surface. When the surface is a topological disk spanning a circle, we may get something more. Our first interest focuses on the area of the surface. We shall prove that a certain control of the area of the surface shows the umbilicity of the surface. In addition, we shall prove that under the assumption of stability, a cmc disk spanning a circle must be umbilical. In particular, this gives a partial positive answer to Conjecture 3. Finally, we end the chapter with estimates of the area of a cmc disk with circular boundary. The results in this chapter appear in [ALP99, AP01, LM95, LM96].

7.1 Disks with Small Area

Consider a compact surface M with the topology of a closed disk. We abbreviate this by saying that M is a disk. The main result in this section is motivated by the isoperimetric inequality (6.7) which we now return to in full generality [BC78].

Lemma 7.1.1 *Consider a disk M with a Riemannian metric ds^2. Let $c \in \mathbb{R}$. If L and A are the length of ∂M and the area of M, respectively, then*

$$L^2 - 2A\left(2\pi - \int_M (K-c)^+ \, dM\right) + cA^2 \geq 0,$$

where K is the Gauss curvature and $(K-c)^+$ is the positive function of $K-c$. Moreover, equality holds if and only if (M, ds^2) is a geodesic disk in a sphere with Gauss curvature c.

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_7,
© Springer-Verlag Berlin Heidelberg 2013

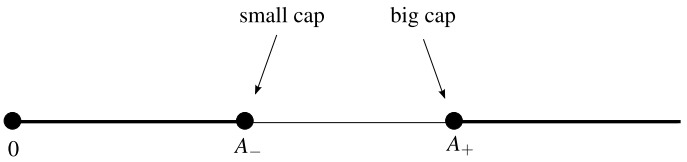

Fig. 7.1 The *bold line* indicates the range of the area of an H-disk with circular boundary. The numbers A_- and A_+ are the bounds of the area A that appear in (7.2). If A coincides with the value A_- or A_+, the surface is a spherical cap. Theorem 7.1.2 proves that the interval $(0, A_-]$ is impossible and it must reduce to the single point $\{A_-\}$

We apply this lemma in the context of cmc surfaces. Let $x : M \to \mathbb{R}^3$ be an immersion of a disk M with constant mean curvature H where, as usual, we are considering on M the induced metric from $\mathbb{R}^3$. Since $K \leq H^2$ and H is constant, we choose $c = H^2$ in the preceding lemma. Then

$$L^2 - 4\pi A + H^2 A^2 \geq 0 \tag{7.1}$$

and equality holds if and only if the surface is umbilical. In this case, the surface is part of a plane if $H = 0$ and part of a sphere of radius $1/|H|$ if $H \neq 0$. We draw attention to the fact that if $H = 0$, then (7.1) asserts $L^2 \geq 4\pi A$, which extends to minimal surfaces the classical isoperimetric inequality of the plane.

Assume that the boundary $\Gamma = x(\partial M)$ is a circle of radius $r > 0$. Then $L = 2\pi r$ and (7.1) becomes

$$4\pi^2 r^2 - 4\pi A + H^2 A^2 \geq 0.$$

Suppose $H \neq 0$. From this quadratic inequality, there are two positive numbers $A_- \leq A_+$ such that

$$A \leq A_- = \frac{2\pi}{H^2}\big(1 - \sqrt{1 - r^2H^2}\big) \quad \text{or} \quad A \geq A_+ = \frac{2\pi}{H^2}\big(1 + \sqrt{1 - r^2H^2}\big). \tag{7.2}$$

Recall that the mean curvature H satisfies $|H| \leq 1/r$ (Corollary 5.1.8). Furthermore, equalities in (7.2) hold if and only if the immersion is umbilical: see the discussion in Sect. 6.2. More precisely, A_- (resp. A_+) is the area of the small spherical H-cap (resp. large spherical H-cap) spanning Γ. See Fig. 7.1.

Theorem 7.1.2 *Let M be a disk and let $x : M \to \mathbb{R}^3$ be an immersion with constant mean curvature $H \neq 0$ and spanning a circle Γ of radius $r > 0$. If*

$$A \leq \frac{2\pi}{H^2}\big(1 + \sqrt{1 - r^2H^2}\big), \tag{7.3}$$

then $x(M)$ is a spherical cap.

Proof Suppose that Γ lies in the plane $z = 0$ and is centered at the origin and let $e_3 = (0, 0, 1)$. Consider a parametrization $\alpha = \alpha(s)$ of Γ by the arc length such that

$\alpha(s) \times \alpha'(s) = re_3$. Let κ_g be the geodesic curvature of ∂M. Since the curvature of Γ is $1/r$, we have

$$\kappa_n^2(s) + \kappa_g^2(s) = \frac{1}{r^2}, \tag{7.4}$$

where $\kappa_n(s) = \sigma(\alpha'(s), \alpha'(s))$ is the normal curvature of ∂M in M. If N is the Gauss map and since $\alpha''(s) = -\alpha(s)/r^2$, we find that

$$\begin{aligned}\kappa_n(s) &= -\langle N'(s), \alpha'(s)\rangle = \langle N(s), \alpha''(s)\rangle = -\frac{1}{r^2}\langle N(s), \alpha(s)\rangle \\ &= -\frac{1}{r}\langle N(s) \times \alpha'(s), e_3\rangle.\end{aligned}$$

The flux formula (5.1) gives

$$\int_{\partial M} \kappa_n(s)\, ds = \frac{H}{r} \int_{\partial M} \langle \alpha(s) \times \alpha'(s), e_3\rangle\, ds = 2\pi r H.$$

We use the Cauchy-Schwarz inequality twice in this relation together with (7.4):

$$\begin{aligned}4\pi^2 r^2 H^2 &= \left(\int_{\partial M} \kappa_n(s)\, ds\right)^2 \le 2\pi r \int_{\partial M} \kappa_n^2(s)\, ds = 4\pi^2 - 2\pi r \int_{\partial M} \kappa_g^2(s)\, ds \\ &\le 4\pi^2 - \left(\int_{\partial M} \kappa_g(s)\, ds\right)^2.\end{aligned}$$

We obtain

$$\left|\int_{\partial M} \kappa_g(s)\, ds\right| \le 2\pi\sqrt{1 - r^2 H^2}. \tag{7.5}$$

As the topology of the surface is a disk, the Gauss-Bonnet theorem says

$$2\pi = \int_M K\, dM + \int_{\partial M} \kappa_g(s)\, ds.$$

Taking into account (7.5) and $K \le H^2$, we deduce

$$2\pi \le AH^2 + 2\pi\sqrt{1 - r^2 H^2}.$$

Hence $A \ge A_-$. In view of (7.2) and (7.3), we conclude $A = A_-$ or A_+, which means that the immersion is umbilical and so, $x(M)$ is a spherical cap. □

Combining Theorem 7.1.2 and the Minkowski formula (6.11), we obtain:

Theorem 7.1.3 *Let Γ be a circle of radius $r > 0$. Consider an immersion $x : M \to \mathbb{R}^3$ of a disk M spanning Γ and with constant mean curvature $H \ne 0$. If the volume V of M satisfies $|V| \le 2\pi r^3/3$, then $x(M)$ is a spherical cap.*

Proof Let a be a unit vector orthogonal to the plane containing Γ. Assume $H > 0$ and let N be the corresponding orientation. From Corollary 6.3.2, $V \geq 0$. If ν is the inner conormal vector of M along ∂M, the Cauchy-Schwarz inequality yields

$$\left(\int_{\partial M} \langle \nu, x\rangle \, ds\right)^2 \leq 2\pi r \int_{\partial M} \langle \nu, x\rangle^2 \, ds = 2\pi r^3 \int_{\partial M} \left(1 - \langle \nu, a\rangle^2\right) ds. \tag{7.6}$$

Using the Cauchy-Schwarz inequality again and the flux formula (5.2), we have

$$\int_{\partial M} \langle \nu, a\rangle^2 \, ds \geq \frac{1}{2\pi r}\left(\int_{\partial M} \langle \nu, a\rangle \, ds\right)^2 = 2\pi r H^2.$$

Substituting into (7.6), we obtain

$$\left|\int_{\partial M} \langle \nu, x\rangle \, ds\right| \leq 2\pi r^2 \sqrt{1 - r^2 H^2}.$$

This inequality provides an upper bound for the right-hand side in the Minkowski formula (6.11),

$$A - 3HV \leq \pi r^2 \sqrt{1 - r^2 H^2}. \tag{7.7}$$

By Theorem 7.1.2, if $x(M)$ is not a spherical cap, its area A satisfies

$$A > A_+ = \frac{2\pi}{H^2}\left(1 + \sqrt{1 - r^2 H^2}\right).$$

From this inequality and (7.7), we see that if $x(M)$ is not a spherical cap, the volume V satisfies

$$V > \frac{2\pi}{3H^3}\left(1 + \sqrt{1 - r^2 H^2}\right) - \frac{\pi r^2}{3H}\sqrt{1 - r^2 H^2}.$$

The right-hand side in this inequality is a decreasing function of H. As the mean curvature H moves in the range $0 < H \leq 1/r$, the minimum occurs at $H = 1/r$, which is $2\pi r^3/3$. This concludes the proof. □

Corollary 7.1.4 *Let Γ be a circle. Then the volume of a non-spherical cmc disk in Euclidean space spanning Γ is greater than the volume of a hemisphere with the same boundary.*

Theorems 7.1.2 and 7.1.3 are related to Corollary 6.2.3, where the hypothesis was $AH^2 \leq 2\pi$. All these results say that if the area or the volume of a compact cmc surface with circular boundary is small in some sense, then the surface is spherical. Whereas in Chap. 6 the assumption was the embeddedness of the surface, here we suppose that the topology of the surface is a disk.

7.2 Stable cmc Disks

In this section we study the stability problem of a compact cmc surface with circular boundary. Consider an immersion $x : M \to \mathbb{R}^3$ of a compact surface M with $\partial M \neq \emptyset$. Define the function

$$\psi : M \to \mathbb{R}, \qquad \psi = \langle x \times N, a \rangle,$$

where $a \in \mathbb{R}^3$. The function ψ represents the normal component of the Killing field $Y(p) = p \times a$ of $\mathbb{R}^3$ corresponding to the rotations around the vector a. For each $p \in M$ and $u \in T_pM$, we have

$$(d\psi)_p(u) = \big\langle (dx)_p(u) \times N(p), a \big\rangle - \big\langle x(p) \times A_p u, a \big\rangle,$$

where A is the Weingarten map of the immersion x. Hence we compute the Hessian $\nabla^2\psi$: if $u, v \in T_pM$

$$\begin{aligned}\left(\nabla^2\psi\right)_p(u,v) = &-\big\langle (dx)_p(v) \times A_p u, a \big\rangle - \big\langle (dx)_p(u) \times A_p v, a \big\rangle \\ &- \big\langle x(p) \times (\nabla A)_p(u,v), a \big\rangle - \langle A_p u, A_p v \rangle \psi(p),\end{aligned}$$

where ∇A is the covariant derivative of A. Taking the trace in this equality, we deduce that

$$\Delta\psi = -2\langle x \times \nabla H, a \rangle - |\sigma|^2 \psi.$$

If the mean curvature is constant, $\nabla H = 0$, and the function ψ satisfies the Jacobi equation.

Lemma 7.2.1 *Let M be a compact surface. Let $x : M \to \mathbb{R}^3$ be an immersion of constant mean curvature and let N be the Gauss map. If $a \in \mathbb{R}^3$, the function $\psi = \langle x \times N, a \rangle$ satisfies*

$$\Delta\psi + |\sigma|^2\psi = 0.$$

If $x(\partial M)$ is a circle in a plane P and a is a vector orthogonal to P, then

$$\int_M \psi \, dM = 0, \quad \psi = 0 \text{ on } \partial M.$$

If $\psi = 0$, the surface is rotationally symmetric with respect to the axis a.

Proof We prove that $\int_M \psi \, dM = 0$. Let $\Omega \subset P$ be the disk bounded by the circle. As the divergence of the vector field $Y(p) = p \times a$ is zero and the normal vector to Ω is $N_\Omega = \pm a/|a|$, we have

$$0 = \int_M \langle x \times a, N \rangle \, dM + \int_\Omega \langle x \times a, N_\Omega \rangle \, dM = \int_M \langle x \times a, N \rangle \, dM. \qquad \square$$

Since ψ is an eigenfunction of the Jacobi operator $L = \Delta + |\sigma|^2$ for $\lambda = 0$, the nodal lines of ψ are piecewise smooth curves [Che76]. If $x(\partial M)$ is a circle and a is a vector orthogonal to the plane containing $x(\partial M)$, the boundary curve ∂M is itself a nodal line. The function ψ allows us to prove:

Theorem 7.2.2 *Planar disks and small spherical caps are the only strongly stable cmc surfaces with circular boundary.*

Proof Consider the function ψ, where a is a unit vector orthogonal to the plane containing the circle. If $\psi = 0$ on M, then M is umbilical. In such case, the surface is a planar disk or a spherical cap. For spherical caps, we know that only the small ones are strongly stable (Proposition 2.1.13).

For the converse, if M is strongly stable, $\lambda_1(L) \geq 0$ by Corollary 2.1.14. Assume that ψ is not constantly zero on M. Then Lemma 7.2.1 says that ψ is an eigenfunction of L with eigenvalue $\lambda = 0$. As a consequence, $\lambda_1(L) = 0$. By the Courant theorem [CH89], ψ does not change sign, a contradiction because $\int_M \psi \, dM = 0$. □

We now state and prove the main result of this section.

Theorem 7.2.3 *Planar disks and spherical caps are the only stable cmc disks with circular boundary.*

In order to emphasize the importance of this result, we recall the three conjectures concerning compact cmc surfaces bounded by a circle established in Chap. 1 assuming topology, embeddedness and stability. Theorem 7.2.3 states that if the surface is a disk and stable, it must be umbilical. We point out that if the surface is embedded and a disk, or embedded and stable, the corresponding questions currently remain unsolved.

Proof Consider conformal coordinates on the surface. Denote by $D = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disk in the complex plane $\mathbb{C}$ and let $x : D \to \mathbb{R}^3$ be a stable immersion with constant mean curvature H with $x(\partial D) = \mathbb{S}^1$, where $\mathbb{S}^1$ is the circle $\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. Let $ds^2 = E|dz|^2$ be the induced metric on M. In addition, we introduce polar coordinates r, θ, with $z = re^{i\theta} = r(\cos\theta + i \sin\theta)$. The unit tangent and unit normal vectors to ∂D with respect to ds^2 are:

$$\mathbf{t} = \frac{1}{\sqrt{E}}\partial_\theta, \qquad \mathbf{n} = \frac{1}{\sqrt{E}}\partial_r.$$

Choose N so that along the boundary it satisfies $dx(\mathbf{t}) \times dx(\mathbf{n}) = N$. Denote by a a unit vector orthogonal to the boundary plane such that $x \times dx(\mathbf{t}) = a$ along ∂D. Take $\psi = \langle x \times N, a \rangle$. The normal derivative of ψ on the boundary is

$$\partial_n \psi = \langle \nabla \psi, \mathbf{n} \rangle = \big\langle dx(\mathbf{n}) \times N, a \big\rangle + \big\langle x \times dN(\mathbf{n}), a \big\rangle.$$

As $dx(\mathbf{n}) \times N = \nu \times N = dx(\mathbf{t})$ and $dN(\mathbf{n}) = -\sigma(\mathbf{t}, \mathbf{n})dx(t) - \sigma(\mathbf{n}, \mathbf{n})dx(\mathbf{n})$, we have

$$\partial_n \psi = -\sigma(\mathbf{t}, \mathbf{n}),$$

where σ is the second fundamental form of the immersion. On a neighborhood of a point on ∂D, choose a branch of the logarithm and make the holomorphic change of coordinates $w = \log z$. Since $w = \log|z| + i\theta \pmod{2\pi i}$, we have

$$\partial_w = \frac{1}{2}(\partial_r - i\partial_\theta)$$

globally on ∂D since ∂_θ is independent of the branch of logarithm. Also, $\partial_w = z\partial_z$ globally on $0 < |z| \leq 1$.

Consider the Hopf differential Φdz^2 on M. From (2.18) we have on $0 < |z| \leq 1$,

$$z^2\Phi = 2z^2\sigma(\partial_z, \partial_z) = 2\sigma(\partial_w, \partial_w).$$

Hence along $|z| = 1$ we see that $\mathrm{Im}(z^2\Phi) = -\sigma(\partial_r, \partial_\theta)$. By using the values of ∂_r and ∂_θ, we derive

$$\mathrm{Im}\big(z^2\Phi\big) = E\partial_n\psi \quad \text{on } \partial D. \tag{7.8}$$

Claim *The normal derivative $\partial_n\psi$ vanishes at least three times on ∂D.*

Proof of the claim We argue by contradiction. Suppose that $\partial_n\psi$ vanishes at most at two points of the boundary. At these points, $\mathrm{Im}(z^2\Phi)$ vanishes by (7.8). There is no loss of generality in assuming that there are exactly two distinct points $z_1, z_2 \in \partial D$.

Since Φ has no singularities on ∂D, we extend Φ analytically to a slightly larger disk of radius $R > 1$. As the zeroes of Φ are isolated points, let $\varepsilon > 0$ be a sufficiently small number so that $\Phi^{-1}(\{0\}) \cap \overline{D_\varepsilon}(z_j) = \{z_j\}$, where $D_\varepsilon(z_j) = \{z \in \mathbb{C} : |z - z_j| < \varepsilon\}$. Let $D_\varepsilon^* = D - (D_\varepsilon(z_1) \cup D_\varepsilon(z_2))$ and let $C_\varepsilon = \partial D_\varepsilon^*$ be oriented positively. Compute the total variation of $\arg(z^2\Phi)$ along the curve C_ε. Since there are no zeros of $z^2\Phi$ on C_ε, we have

$$\frac{1}{i}\int_{C_\varepsilon}\left(\frac{2}{z} + \frac{\Phi'}{\Phi}\right)dz = 4\pi + 2\pi m(\varepsilon), \tag{7.9}$$

where $m(\varepsilon)$ is the number of zeros inside C_ε with multiplicity. Calculating residues, we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{i}\int_{C_\varepsilon \cap \partial D_\varepsilon(z_j)}\frac{\Phi'}{\Phi}\,dz = -n_j\pi,$$

where n_j is the order of the zero of Φ at z_j. Then

$$\frac{1}{i}\lim_{\varepsilon\to 0}\int_{C_\varepsilon \setminus \partial(D_\varepsilon(z_1)\cup D_\varepsilon(z_2))}\left(\frac{2}{z} + \frac{\Phi'}{\Phi}\right)dz = 4\pi + 2\pi m + \pi(n_1 + n_2),$$

where m denotes the number of umbilical points, with multiplicities in the interior of D.

If $n_1 = n_2 = 0$, the image of ∂D under $z^2\Phi$ has winding number at least 2 with respect to the origin which implies that $\mathrm{Im}(z^2\Phi)$ vanishes at least four distinct points on ∂D. If $n_1 + n_2 > 0$, by the last equation, the total variation of $\arg(z^2\Phi)$ over the two arcs determined by z_1 and z_2 is at least 5π. Hence, there must be at least three points on ∂D where $\mathrm{Im}(z^2\Phi)$ vanishes. This ends the proof of the claim. □

Now we prove the theorem. Proceeding as in the proof of Theorem 7.2.2, either M is umbilical or the function ψ changes sign. In view of Lemma 7.2.1, $\lambda = 0$ is an eigenvalue of the Jacobi operator in (2.12), and since ψ changes sign, $\lambda_1 \neq 0$ and so, $\lambda_1 < 0$.

We know that the k-th eigenfunction f_k of L divides D at most into k nodal domains [CH89]. Let f_2 be an eigenfunction corresponding to λ_2. Assume that $\partial_n f_2$ vanishes at a boundary point $p \in \partial D$. As $f_2 = 0$ along ∂D, we have $\nabla f_2(p) = 0$. We show that there is a nodal line $\mathcal{L}$ distinct from ∂D ending at p. Assuming the contrary, there is an open domain U around p such that f_2 has sign, except at $U \cap \partial D$. Assume, for example, that $f_2 > 0$ on U. Then on U, we have

$$\Delta f_2 = -|\sigma|^2 f_2 \leq 0$$

and the maximum principle implies that f_2 assumes its maximum at the boundary. Thus, $f_2 \leq \max_{\partial U} f_2 = 0$. This contradiction proves the existence of $\mathcal{L}$.

Since f_2 has at most two nodal lines, the derivative $\partial_n f_2$ vanishes at most two times at ∂D. By the claim, $\psi = f_n$ for some $n \geq 3$. Therefore $\lambda_2 < 0$. Define

$$c = -\frac{\int_D f_2 dD}{\int_D f_1 dD}$$

and let $f = f_2 + cf_1 \in H_0^{1,2}(M)$. Then $f = 0$ along ∂D and $\int_D f dD = 0$. However, using that $L(f_i) + \lambda_i f_i = 0$ and that $\int_D f_i f_j dD = 0$, we find that

$$I(f) = \lambda_2 \int_D f_2^2 dD + c^2 \lambda_1 \int_D f_1^2 dD < 0,$$

which is not possible by the stability of the surface. This shows that $\psi = 0$ on D. It follows that the surface is rotational and hence the surface is a planar disk or a spherical cap. This concludes the proof of theorem. □

Remark 7.2.4 Similar arguments as in Theorem 7.2.3 have been used for stable immersions from a disk in Euclidean space spanning a circle and minimizing other functionals different from the surface area, such as the anisotropic and the Willmore energies: see the works of M. Koiso and B. Palmer [KP13, Pal00].

7.3 Other Area Estimates of a cmc Disk

In Theorem 7.1.2 we obtained an estimate of the area of a non-umbilical cmc disk bounded by a circle. The purpose of this section is to find other area estimates using the properties of the function $\psi = \langle x \times N, a \rangle$ and Theorem 7.2.3. First we need the following result, which appears in [Ruc79] (see also [HO82]).

Lemma 7.3.1 *Let $x : M \to \mathbb{R}^3$ be an immersion with constant mean curvature H. If ds^2 is the induced metric on M, define the metric*

$$d\overline{s}^2 = \frac{|\sigma|^2}{2} ds^2 = \big(2H^2 - K\big) ds^2.$$

Then the Gaussian curvature $\overline{K}$ of the Riemannian surface $(M, d\overline{s}^2)$ satisfies $\overline{K} \leq 1$ and equality holds if and only if the immersion is umbilical.

The relation between Δ and the Laplacian $\overline{\Delta}$ computed with respect to the metric $d\overline{s}^2$ is $\overline{\Delta} = 2/|\sigma|^2 \Delta$. From Lemma 7.2.1, we have

$$\overline{\Delta}\psi + 2\psi = 0. \tag{7.10}$$

Theorem 7.3.2 *Let $H \neq 0$ and let M be a topological disk. Let $x : M \to \mathbb{R}^3$ be an immersion with constant mean curvature H and spanning a circle Γ of radius $r > 0$. If the area A of M satisfies*

$$A < \frac{4\pi}{H^2} - \frac{\pi}{H^2}\sqrt{1 - r^2 H^2}, \tag{7.11}$$

then $x(M)$ is a spherical cap.

Proof Let us assume for contradiction that the surface is not spherical. Define the function ψ taking the vector a as a unit vector orthogonal to the plane containing the circle Γ. From the proof of Theorem 7.2.3 we know that the normal derivative $\partial_n \psi$ must vanish at least three times on the boundary of the surface. Because ψ must change sign across each nodal curve interior to M and since M is a topological disk, there are at least three simply connected nodal domains, D_i, $i = 1, 2, 3$. Since ψ does not vanish in each D_i, $\psi_i = \psi_{|D_i}$ is the first eigenfunction of the Laplacian $\overline{\Delta}$ with Dirichlet condition in D_i and the first eigenvalue is 2 by (7.10).

We use the Faber-Krahn inequality [BC80, Proposition 3.3]. As $\overline{K} \leq 1$, we compare the first eigenvalue of the Dirichlet problem with that of domains of a sphere of curvature 1, say, of the unit sphere $\mathbb{S}^2$. Denote by $\overline{A_i}$ the area of $(D_i, d\overline{s}^2)$. Assume that $\overline{A_i} < 2\pi$ and let D_i' be a geodesic disk $D_i' \subset \mathbb{S}^2$ with $\text{area}(D_i') = \overline{A_i}$. Then the Faber-Krahn inequality asserts that $\lambda_1(D_i') \leq \lambda_1(D_i, d\overline{s}^2)$. Consequently, $\lambda_1(D_i') \leq 2$. However, proper inclusions strictly increase the first eigenvalue [CH89]. As $\text{area}(D_i') < 2\pi$, D_i' is included strictly in a hemisphere of

$\mathbb{S}^2$ and the first eigenvalue of a hemisphere is 2. We would have $\lambda_1(D_i') > 2$, a contradiction. Therefore, $\overline{A_i} \geq 2\pi$ for all $i = 1, 2, 3$.

By the definition of $d\overline{s}^2$ and if $\overline{A}$ is the area of $(M, d\overline{s}^2)$, we have

$$\overline{A} = \int_M \left(2H^2 - K\right) dM = 2H^2 A - \int_M K \, dM.$$

As $\overline{A} \geq \sum_{i=1}^3 \overline{A_i} \geq 6\pi$, the above inequality yields

$$A \geq \frac{3\pi}{H^2} + \frac{1}{2H^2} \int_M K \, dM. \tag{7.12}$$

From (7.5) and the Gauss-Bonnet theorem,

$$2\pi = \int_M K \, dM + \int_{\partial M} \kappa_g \, ds \leq \int_M K \, dM + 2\pi\sqrt{1 - r^2 H^2}.$$

Substituting into (7.12), the proof is now complete. □

We compare estimates (7.3) and (7.11). Let

$$A_2 = \frac{4\pi}{H^2} - \frac{\pi}{H^2}\sqrt{1 - r^2 H^2},$$

the right-hand side in (7.11). Then

$$A_2 < A_+ \quad \text{when} \quad 0 < H^2 < \frac{5}{9r^2},$$

and

$$A_2 > A_+ \quad \text{when} \quad \frac{5}{9r^2} < H^2 \leq \frac{1}{r^2}.$$

Remark 7.3.3 Theorem 7.3.2 improves a result given in [Lop99a], which states that, under the same assumption, if $\int_M |\sigma|^2 \, dM \leq 8\pi$, then M is a flat disk or a spherical cap.

Chapter 8
The Dirichlet Problem of the cmc Equation

In this and the next chapter we shall study the Dirichlet problem for the cmc equation assuming in the present chapter that the domain is bounded. As we shall see, the existence of a solution to the Dirichlet problem reduces to the question of obtaining a priori estimates for the solution and its gradient. This is possible thanks to the type of second order equation and the topological apparatus of a priori estimates described by Ladyzhenskaya and Uraltseva. For a systematic introduction to the Dirichlet problem for elliptic equations of second order, including results of regularity, we refer the reader to Gilbarg-Trudinger [GT01]. Other references include [Bak94, DHKW92, DHS10, Sau06, Sim97].

The purpose of the present chapter is to study the geometry that lies behind the Dirichlet problem, giving a geometric interpretation of the usual techniques that appear in the theory of PDEs. In this sense we skip the discussion of regularity and we will assume that the initial data in the Dirichlet problem is sufficiently smooth to guarantee that the solutions are smooth. We shall make geometric assumptions on the length or the curvature of the boundary curve and we shall use known surfaces such as pieces of Delaunay surfaces as barriers to control the graphs that are solutions of the Dirichlet problem. If these barriers are spherical caps or cylinders, we shall assume that the domain is convex. When we employ nodoids, we suppose that the domain satisfies a certain exterior circle condition. The reason for focussing on convex domains is not only because they are easy to work with but also because if the Dirichlet problem is solvable for arbitrary data, the domain must be strictly convex. We finish the chapter with a discussion of the corresponding Dirichlet problem for the cmc equation in domains of the unit sphere $\mathbb{S}^2$ whose solutions represent radial graphs.

From the viewpoint of the theory of PDEs the mean curvature equation falls into the family of elliptic equations of divergence type, as was remarked in Chap. 3. The literature for this equation is very extensive where the constant H in the mean curvature equation is substituted by a function $f = f(x, \nabla u)$. In differential geometry, the Dirichlet problem has received interest in recent years, see for example, [Bgn09, Daj09, DR11, EsRi01, FLR02, Lop99b, Lop01b, Lop02a, LM96, Rip01]. On the other hand, results of existence have been extended to other ambient spaces

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_8,
© Springer-Verlag Berlin Heidelberg 2013

and contexts: [AD07, ADR07, DHL08, DL09, DR05, ET11, EFR10, FM11, FR12, HRS09, MRR11, Pin09]. See Chap. 11 for a treatment of the Dirichlet problem in hyperbolic space and Sect. 12.6 in Lorentz-Minkowski space.

8.1 The Method of Continuity

Let $\Omega \subset \mathbb{R}^2$ be a smooth domain, $H \in \mathbb{R}$ and $\varphi \in C^0(\partial\Omega)$. The Dirichlet problemfor the mean curvature equation consists of finding a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the boundary value problem

$$Q_H(u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - 2H = 0 \quad \text{in } \Omega \tag{8.1}$$

$$u = \varphi \quad \text{on } \partial\Omega. \tag{8.2}$$

The geometric meaning of (8.1)–(8.2) is that the graph of a solution u describes a non-parametric surface M spanning the curve $\Gamma = \operatorname{graph}(\varphi) = \{(q, \varphi(q)) : q \in \partial\Omega\}$ and the mean curvature of M is H, computed with respect to the upwards orientation

$$N\big(x, y, u(x, y)\big) = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}(x, y). \tag{8.3}$$

We say that M is an *H-graph*. Denote by $P(H, \varphi)$ the problem (8.1)–(8.2).

Although in the discussion of the results we shall require, if necessary, sufficient smoothness of Ω and φ, it may occur that the gradient of the solution blows-up near the boundary points of Ω, and the solution may not be smooth on $\partial\Omega$. For example, take the unit disk $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ and $\varphi = 0$ on $\partial\Omega$. For $H = 1$, the solution of (8.1)–(8.2) is the function $u(x, y) = -\sqrt{1 - x^2 - y^2}$ and the graph of u represents a hemisphere of radius 1. In this case, $|\nabla u| = \infty$ on $\partial\Omega$ and u only extends continuously to $\overline{\Omega}$.

In this chapter, we will identify the Euclidean plane $\mathbb{R}^2$ with the plane P of $\mathbb{R}^3$ with equation $z = 0$. Let

$$e_3 = (0, 0, 1).$$

The technique that we employ to solve (8.1)–(8.2) is the so-called *method of continuity*. For details, we refer the reader to [GT01, Sects. 11.4, 17.2]. The idea is that we 'continuously deform' the problem $P(H, \varphi)$ by defining a 1-parameter family of Dirichlet problems in such a way that at a starting time, the problem can be solved and a further analysis of the curve of solutions together with an argument on connectedness allows us to complete the set of parameters, solving the initial Dirichlet problem $P(H, \varphi)$. We shall see that this process reduces to estimating $|u|$ and $|\nabla u|$ on Ω for a solution u of (8.1)–(8.2).

We present this technique in our context. Assume that φ is smooth on $\partial\Omega$. Consider the set of Dirichlet problems $P(tH, t\varphi)$ parameterized by $t \in [0, 1]$:

$$Q_{tH}(u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - 2tH = 0 \quad \text{in } \Omega$$
$$u = t\varphi \quad \text{on } \partial\Omega.$$

Define the set

$$\mathscr{S} = \left\{t \in [0,1] : \text{there exists a solution } u_t \text{ of } P(tH, t\varphi)\right\}.$$

In order to obtain a solution of $P(H, \varphi)$, we need to show that $\mathscr{S}$ is a non-empty open and closed subset of $[0, 1]$ and, by connectedness, $\mathscr{S} = [0, 1]$. In particular, $1 \in \mathscr{S}$ and this solves $P(H, \varphi)$. We will prove that $\mathscr{S}$ is always a non-empty and open subset of $[0, 1]$, and we will establish conditions to guarantee that $\mathscr{S}$ is closed.

The starting point of the continuity method is the existence of some $t \in \mathscr{S}$ and in this case, it corresponds with the value $t = 0$. Indeed, the function $u_0 = 0$ is a solution of $P(0, 0)$ and thus $0 \in \mathscr{S}$ and $\mathscr{S} \neq \emptyset$.

Now that the existence of the solution u_0 has been established, we are going to show that $\mathscr{S}$ is an open set in $[0, 1]$.

Proposition 8.1.1 *The set $\mathscr{S}$ is open in* $[0, 1]$.

Proof Let $t_0 \in \mathscr{S}$. We show that there exists an $\varepsilon > 0$ such that $(t_0 - \varepsilon, t_0 + \varepsilon) \cap [0, 1] \subset \mathscr{S}$, that is, for any t in such interval, there exists a solution u_t of $P(tH, t\varphi)$.

Consider a more general context. Let $x : M \to \mathbb{R}^3$ be an isometric immersion of a compact surface M and let N be a fixed orientation on M. For each $u \in C_0^{2,\alpha}(M)$, define the admissible normal variation $x_t : M \to \mathbb{R}^3$ by $x_t(p) = x(p) + tu(p)N(p),\ p \in M, t \in (-\varepsilon, \varepsilon)$. Let $H(t)$ be the mean curvature of x_t.

Lemma 8.1.2 *Up to constants, the derivative $H'(0)$ is*

$$H'(0) = \frac{d}{dt}\bigg|_{t=0} H\big(x_t(p)\big) = L(u) = \Delta u + |\sigma|^2 u, \tag{8.4}$$

where Δ denotes the Laplacian operator on M and σ is the second fundamental form of the immersion x.

Proof See (A.10), Appendix A. □

The operator $L = \Delta + |\sigma|^2$ is the Jacobi operator that we met earlier in the stability theory (see Eq. (2.11)). This operator is elliptic and has the following properties [CH89]:

Lemma 8.1.3 *Let $\lambda \in \mathbb{R}$. Consider the eigenvalue problem*

$$\begin{aligned} Lu + \lambda u &= 0 \quad \text{on } M \\ u &= 0 \quad \text{on } \partial M, \end{aligned}$$

The eigenvalues of L form a discrete set $\lambda_1 < \lambda_2 \leq \cdots$ with $\lim_{i\to\infty} \lambda_i = +\infty$. Let V_{λ_i} denote the eigenspace subspace of λ_i.

Let $h \in C_0^\infty(M)$. If λ is not an eigenvalue of L, there exists a solution of the equation $Lu + \lambda u = h$. If λ_i is an eigenvalue of L, there exists a solution of $Lu + \lambda_i u = h$ if and only if h is L^2-orthogonal to V_{λ_j} for $j < i$.

We return to the proof of Proposition 8.1.1. Let $t \in \mathscr{S}$. Define the function

$$F : C_0^{2,\alpha}(\overline{\Omega}) \times [0,1] \to C_0^\alpha(\overline{\Omega}), \qquad F(v,t) = Q_{tH}(v + t\varphi),$$

where with by same symbol φ we denote a $C^{2,\alpha}$-extension to Ω of the initial function φ. Let $u_t = v + t\varphi$ and let x be the immersion of the graph of u_t. As $t \in \mathscr{S}$, we have $Q_{tH}(u_t) = 0$ and $F(u_t - t\varphi, t) = 0$. Given $w \in C_0^2(\overline{\Omega})$, define $x_s(p) = x(p) + sw(p)N(p)$ and let $H(s)$ be the mean curvature of the immersion x_s. Then

$$F(v + sw, t) = \operatorname{div} \frac{\nabla(v + sw + t\varphi)}{\sqrt{1 + |(v + sw + t\varphi)|^2}} - 2tH = 2\big(H(s) - H(0)\big).$$

We compute $(D_1 F)_{(v,t)}(w)$. From the expression of $H'(0)$ we have

$$(D_1 F)_{(v,t)}(w) = 2H'(0) = \Delta w + |\sigma|^2 w,$$

where Δ and σ are respectively the Laplacian operator and the second fundamental form of the immersion x_t.

Lemma 8.1.4 *The operator $(D_1 F)_{(v,t)} = L$ is invertible.*

Proof

1. The operator L is injective. Let $w \in C_0^{2,\alpha}(\overline{\Omega})$ be a function such that $\Delta w + |\sigma|^2 w = 0$. Assume $w \neq 0$. As $w = 0$ on $\partial\Omega$, $\lambda = 0$ is an eigenvalue of L. On the other hand, the graph of u_t is strongly stable and thus $\lambda_1(L) > 0$ by Lemma 2.3.9, obtaining a contradiction.
2. The operator L is surjective. By Lemma 2.3.9, $\lambda_1(L) > 0$. Then 0 is not an eigenvalue of L. Using Lemma 8.1.3 and given $h \in C_0^\alpha(\overline{\Omega})$, there exists a $w \in C_0^{2,\alpha}(\overline{\Omega})$ such that $\Delta w + |\sigma|^2 w = h$, i.e. $Lw = h$.

□

Finally, the invertibility of L concludes the proof of openness by the implicit function theorem. Indeed, if $t_0 \in \mathscr{S}$, there exists an $\varepsilon > 0$, an open set V around

$u_{t_0} - t_0\varphi$ and a diffeomorphism $h : (t_0 - \varepsilon, t_0 + \varepsilon) \to V$ such that for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, $F(h(t), t) = 0$. If we let $u_t = h(t) + t\varphi$, then u_t is a solution of $P(tH, t\varphi)$. This proves Proposition 8.1.1. □

Remark 8.1.5 From the proof, the invertibility of L has been proved thanks to the strong stability of an H-graph and the standard properties of the eigenvalues of an elliptic operator. The first part was proved in Lemma 2.3.9 via the existence of a function $g < 0$ such that $Lg \geq 0$. In other words, we have proved that the kernel of L is trivial and so, L is a self-adjoint Fredholm operator of index 0. In fact, as was remarked in Chap. 2, it is possible to perturb a strong stable H-surface by means of a variation by cmc surfaces [Koi02, Tom75].

As a consequence, we have the following result, which we used in previous chapters (e.g., Sects. 5.2 and 6.3).

Corollary 8.1.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then there exists a constant $H_0 = H_0(\Omega) > 0$ depending only on Ω such that if $|H| < H_0$, the problem $P(H, 0)$ has a unique solution.*

Proof Let us take $\varphi = 0$, $H = 1$ in the Dirichlet problem and let $\mathscr{S}$ be the corresponding set. As the function $u = 0$ solves $P(0, 0)$, we have $0 \in \mathscr{S}$. Since 0 is an interior point of $\mathscr{S}$, there exists an $\varepsilon = \varepsilon(\Omega) > 0$ such that $[0, \varepsilon) \subset \mathscr{S}$. This proves the existence of solutions of $P(H, 0)$ for $0 \leq H < \varepsilon$. Choose $H_0 = \varepsilon$. For values H in $(-H_0, 0]$, we make use of Remark 3.2.7. □

Finally, we proceed to study the question of whether $\mathscr{S}$ is a closed set in $[0, 1]$.

Proposition 8.1.7 *Assume that there are a priori $C^{1,\alpha}(\overline{\Omega})$-estimates of solutions of $P(tH, t\varphi)$ for all $t \in [0, 1]$. Then the set $\mathscr{S}$ is closed in $[0, 1]$.*

Proof Consider $\{t_n\} \subset \mathscr{S}$ where $t_n \to t$. Let u_n be the corresponding solutions of $P(t_n H, t_n\varphi)$. As $\{u_n\}$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ and φ and Ω are smooth, the theory of Schauder estimates for quasilinear elliptic equations implies that $\{u_n\}$ is a bounded set in $C^{2,\alpha}(\overline{\Omega})$ [GT01, Theorem 6.6]. By the Ascoli-Arzelá theorem, there is a subsequence (which we also call) u_n that converges uniformly to $u \in C^2(\overline{\Omega})$ in the C^2-norm. On the other hand, the map

$$T : [0, 1] \times C^2(\overline{\Omega}) \to C^0(\overline{\Omega}), \qquad T(u, t) = Q_{tH}(u)$$

is continuous. Since $t_n \to t$ and $u_n \to u$ in $C^2(\overline{\Omega})$, we have

$$Q_{tH}(u) = T\Big(\lim_{n\to\infty}(t_n, u_n)\Big) = \lim_{n\to\infty} T(t_n, u_n) = \lim_{n\to\infty} Q_{t_n H}(u_n) = 0.$$

This proves that $Q_{tH}(u) = 0$ in Ω. Moreover,

$$u_{|\partial\Omega} = \lim_{n\to\infty}(u_{n|\partial\Omega}) = \lim_{n\to\infty}(t_n\varphi) = t\varphi.$$

As φ and $\partial\Omega$ are smooth, it follows that $u \in C^\infty(\overline{\Omega})$ [GT01, Theorem 6.19]. This proves that $t \in \mathscr{S}$. □

From Proposition 8.1.7, in order to establish that $\mathscr{S}$ is a closed set, it suffices to find a priori estimates in $C^{1,\alpha}(\overline{\Omega})$ for a prospective solution of $P(H,\varphi)$. In fact, because the given data Ω and φ in $P(H,\varphi)$ are smooth, we only require $C^1(\overline{\Omega})$ estimates of solutions of (8.1)–(8.2) [GT01, Theorem 13.8].

Thus the question of the solvability of the Dirichlet problem $P(H,\varphi)$ reduces to the problem of controlling $\sup_\Omega |u|$ and $\sup_\Omega |\nabla u|$ for a prospective solution u by a constant that depends only on the initial data of the Dirichlet problem.

Proposition 8.1.8 *In the continuity method, if there exists a constant $C > 0$ independent of t such that for any solution u_t of $P(tH, t\varphi)$ we have*

$$|u_t|_{C^1(\overline{\Omega})} = \sup_{\overline{\Omega}} |u_t| + \sup_{\overline{\Omega}} |\nabla u_t| < C,$$

then $\mathscr{S}$ is a closed set in $[0,1]$.

Remark 8.1.9 If φ is only continuous on $\partial\Omega$, we approximate φ uniformly by smooth functions φ_n. Suppose that we can solve the Dirichlet problems $P(H,\varphi_n)$ obtaining smooth solutions u_n for each n. By Schauder theory, there exists a subsequence (which we also call) u_n that converges locally in $C^k(\Omega)$ to a function u and u solves (8.1) in Ω. In order to prove that u is continuous on $\overline{\Omega}$ and $u = \varphi$ on $\partial\Omega$, we apply the maximum principle to the difference function $u_n - u_m$ as in Lemma 3.2.3. Then

$$\sup_\Omega |u_n - u_m| = \sup_{\partial\Omega} |u_n - u_m| = \sup_{\partial\Omega} |\varphi_n - \varphi_m|.$$

As φ_n is a Cauchy sequence in $C^0(\partial\Omega)$, the same is true for u_n, and this proves that $u_n \to u$ uniformly on $\overline{\Omega}$.

Lemma 8.1.10 *There are always a priori C^0-estimates for u_t.*

Proof If $H = 0$, the graph is included in the convex hull of its boundary $\Gamma = \mathrm{graph}(\varphi)$, giving C^0-estimates depending only on the boundary.

Assume now that $H \neq 0$. As $t = 0$ is an interior point of $\mathscr{S}$, there exists a $t_0 > 0$, $t_0 = t_0(\Omega, \varphi, H)$, such that $P(tH, t\varphi)$ has a solution for $t \in [0, t_0)$. In particular, the solutions u_t satisfy a priori C^0-estimates in the interval $[0, t_0)$. Assume $t \geq t_0$. By Theorem 2.3.5, we find that

$$u_t \leq \max_{\partial\Omega} t\varphi + \frac{1}{t|H|} \leq \max_{\partial\Omega} |\varphi| + \frac{1}{t_0|H|},$$

$$u_t \geq \min_{\partial\Omega} t\varphi - \frac{1}{t|H|} \geq -\max_{\partial\Omega} |\varphi| - \frac{1}{t_0|H|}.$$

It follows that

$$|u_t| \leq \max_{\partial\Omega} |\varphi| + \frac{1}{t_0|H|}$$

for $t \in [t_0, 1]$. This inequality, and the argument given for values t in $[0, t_0]$, yields C^0-estimates of u_t for each $t \in [0, 1]$. □

In order to derive the estimates of $|\nabla u|$ in $\overline{\Omega}$, we now show that the supremum of $|\nabla u|$ is attained at $\partial\Omega$.

Lemma 8.1.11 *Given a solution u of $P(H, \varphi)$, we have*

$$\sup_{\overline{\Omega}} |\nabla u| = \max_{\partial\Omega} |\nabla u|.$$

Proof By the expression of N in (8.3), we know that

$$\langle N, e_3 \rangle = \frac{1}{\sqrt{1 + |\nabla u|^2}}. \tag{8.5}$$

Then there are a priori estimates of $|\nabla u|$ provided $\langle N, e_3 \rangle$ remains bounded away from zero. By Eq. (2.24), $\Delta\langle N, e_3 \rangle \leq 0$ and the maximum principle asserts that $\langle N, e_3 \rangle \geq \min_{\partial\Omega} \langle N, e_3 \rangle$. Then $\inf_{\Omega} \langle N, e_3 \rangle \geq \min_{\partial\Omega} \langle N, e_3 \rangle$, obtaining the desired result. □

Now we shall restrict our considerations to the case where $\varphi = 0$ on $\partial\Omega$. We prove that $C^1(\overline{\Omega})$-estimates of any solution u_t of $P(tH, 0)$ reduce to $P(H, 0)$. Let ν be the inner unit conormal vector of $M = \text{graph}(u)$, where u is the solution of $P(H, 0)$, and let α be a parametrization by the arc length of $\partial\Omega$. The expression of the vector e_3 with respect to the basis $\{\alpha', \nu, N\}$ gives $\langle N, e_3 \rangle^2 + \langle \nu, e_3 \rangle^2 = 1$. According to the choice of the orientation N on M, we have

$$|\nabla u| = \frac{|\langle \nu, e_3 \rangle|}{\sqrt{1 - \langle \nu, e_3 \rangle^2}}. \tag{8.6}$$

As a consequence of this reasoning:

> For the Dirichlet problem $P(H, 0)$, we obtain estimates of $|\nabla u|$ on $\overline{\Omega}$ provided we are able to find a constant $C < 1$, depending only on the initial data, such that $|\langle \nu, e_3 \rangle| \leq C$ on $\partial\Omega$.

We call the *slope of a graph M* at a boundary point p the value $|\langle \nu(p), e_3 \rangle|$.

Lemma 8.1.12 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $q \in \partial\Omega$. Consider $u^-, u, u^+ \in C^2(\overline{\Omega})$ such that*:

1. $u^-(q) = u(q) = u^+(q)$.
2. $u^- \leq u \leq u^+$ *in a neighborhood of q in $\overline{\Omega}$.*

Then

$$|\nabla u|(q) \le \max\big\{\big|\nabla u^-\big|(q), \big|\nabla u^+\big|(q)\big\}.$$

Proof Let $C = \max\{|\nabla u^-|(q), |\nabla u^+|(q)\}$. Let $\eta \in \mathbb{R}^2$ be a unit vector at q pointing inside Ω. Let $\alpha : [0, \varepsilon) \to \overline{\Omega}$ be a curve such that $\alpha(0) = q$ and $\alpha'(0) = \eta$. By hypothesis,

$$u^-\big(\alpha(t)\big) \le u\big(\alpha(t)\big) \le u^+\big(\alpha(t)\big), \quad t \in [0, \varepsilon).$$

By the definition of C, for each $\delta > 0$, there exists an $\varepsilon' = \varepsilon'(\delta) < \varepsilon$ such that

$$\left|\frac{u^-(\alpha(t)) - u(q)}{t}\right|, \left|\frac{u^+(\alpha(t)) - u(q)}{t}\right| \le C + \delta,$$

for $t \in [0, \varepsilon')$. Then we have

$$-C - \delta \le \frac{u^-(\alpha(t)) - u(q)}{t} \le \frac{u(\alpha(t)) - u(q)}{t} \le \frac{u^+(\alpha(t)) - u(q)}{t} \le C + \delta.$$

Letting $t \to 0$, we conclude

$$\big\langle \nabla u(q), \eta \big\rangle \le C + \delta.$$

Since this holds for any $\delta > 0$ and for any η, we have $|\nabla u|(q)| \le C$, proving the result. □

Lemma 8.1.13 *Let u be a solution of $P(H, 0)$. Then for any $t \in [0, 1]$, $|\langle \nu_t(q), e_3\rangle| \le |\langle \nu(q), e_3\rangle|$, $q \in \partial\Omega$, where ν_t (resp. ν) denotes the unit conormal vector of the graph of the solution u_t of $P(tH, 0)$ (resp. u) along the boundary.*

Proof By the comparison principle, we know that $u \le u_t \le u_0 = 0$ (Corollary 3.1.8). Then Lemma 8.1.12 implies $|\nabla u_t|(q) \le |\nabla u|(q)$ for any $q \in \partial\Omega$ and $t \in [0, 1]$. Finally, (8.6) yields $|\langle \nu_t(q), e_3\rangle|(q) \le |\langle \nu(q), e_3\rangle|$. □

In conclusion, we have:

Proposition 8.1.14 *Assume that there exists a bound $C < 1$ depending only on Ω and H, such that the slope of an H-graph on Ω with boundary $\partial\Omega$ is bounded by C. Then the Dirichlet problem $P(H, 0)$ is solvable.*

By the monotonicity on H of the solutions of $P(H, 0)$, we obtain:

Corollary 8.1.15 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. If $P(H, 0)$ has a solution, then $P(h, 0)$ is solvable for any $-|H| \le h \le |H|$.*

Proof We may suppose that $h \ge 0$. Let u_h be a solution of $P(h, 0)$. By Lemma 8.1.13, $|\langle \nu_h, e_3\rangle| \le |\langle \nu_H, e_3\rangle|$, obtaining a priori estimates of the slope of u_h. □

Now we return to a question posed in Chap. 5. Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain. We know that if $P(H,0)$ has a solution, then $|H| < L/(2a(\Omega))$ where L is the length of $\partial\Omega$. However we do not know the range of H for which the Dirichlet problem is solvable. We now assert that the set of such $H's$ is an interval [Mcc99].

Corollary 8.1.16 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then there exists a constant H_{max}, $0 < H_{max} < \infty$ such that for any H with $|H| < H_{max}$, the Dirichlet problem $P(H,0)$ is solvable and if $|H| > H_{max}$, the problem $P(H,0)$ has no solution.*

Proof Let $H_{max} = \sup\{H \in \mathbb{R} : P(H,0) \text{ has a solution}\}$. We know that if $P(H,0)$ has a solution, then $|H| \leq L/2a(\Omega)$ and thus, $H_{max} \leq L/(2a(\Omega))$, which shows $H_{max} < \infty$. On the other hand, by Corollary 8.1.6, $H_{max} > 0$. We now prove that for any $H \in (0, H_{max})$, $P(H,0)$ has a solution. Let H be a number with $0 < H < H_{max}$. By the definition of H_{max}, let $h_0 > 0$ such that $H < h_0 \leq H_{max}$ and $P(h_0,0)$ has a solution. Corollary 8.1.15 ensures that $P(H,0)$ is solvable. □

In the extremal case $H = H_{max}$, take a sequence $H_n \nearrow H_{max}$ with the corresponding solutions u_n of $P(H_n,0)$. Let $u(p) = \lim u_n(p)$, $p \in \Omega$. Then a compactness argument as in Proposition 8.1.7 proves that u solves $P(H_{max},0)$ in Ω. We claim that the function u blows-up at a boundary point, that is,

$$\sup_{p\in\partial\Omega} |\nabla u|(p) = \infty.$$

Assuming the contrary, we have $\sup_{\overline{\Omega}} |\nabla u| < \infty$. Then u extends smoothly to $\overline{\Omega}$ and $u = \lim u_n = 0$ on $\partial\Omega$. Then u is a smooth solution in $\overline{\Omega}$ of the Dirichlet problem $P(H_{max},0)$. Proposition 8.1.1 proves that the Dirichlet problem $P(H,0)$ is solvable for H exceeding the value H_{max}, in contradiction with the definition of H_{max}. We point out that the function u will extend continuously to $\partial\Omega$ if we provide barriers that allow us to control the function u near to $\partial\Omega$.

We finish this section with a new proof of Theorem 4.2.1 which does not use the Alexandrov reflection method but instead solves a Dirichlet problem for the cmc equation [Lop99c].

Proof of Theorem 4.2.1 Let M be a compact embedded H-surface spanning a simple closed curve contained in the plane P with equation $z = 0$. Assume that a neighborhood of ∂M in M is a graph over Ω, where $\Omega \subset P$ is the domain bounded by $\partial M = \partial\Omega$. The proof consists of two steps.

Claim *There exists an H-graph G on Ω with $\partial G = \partial M$.*

In order to prove the claim, let $W \subset \mathbb{R}^3$ be the 3-domain determined by $M \cup (\partial\Omega \times (-\infty, 0])$ that contains Ω and orient M by the Gauss map N such that $H > 0$. With this choice, the orientation N points towards the domain W.

By the continuity method, we require a priori estimates of the slope of G along $\partial\Omega$. After a reflection with respect to P, assume that G lies above P. On G we choose the orientation pointing downwards and thus its mean curvature is positive.

Move G down by vertical translations ϕ_t, where $\phi_t(p) = p + te_3$ with $t < 0$. Let $G_t = \phi_t(G)$. Observe that $int(G_t) \subset W$ for all $t < 0$. For sufficiently large negative t, $G_t \cap M = \emptyset$ and $int(G_t) \subset W$. Next we move G_t up until the first intersection point with M at $t = t_1$. If this occurs at $t_1 < 0$, the contact point must be interior in both surfaces, and the tangency principle would say that $G \subset M$. This however contradicts that $\partial G_{t_1} = \partial\Omega \times \{t_1\}$, $\partial M = \partial\Omega$ and $t_1 < 0$.

We can move G_t up until the initial position $t = 0$. In particular, $G \subset \overline{W}$ and the slope of G along $\partial\Omega$ is less than of M. As M is a graph on Ω around $\partial\Omega$ and if ν and ν_G denote the inward unit conormal vectors of M and G along $\partial\Omega$, respectively, we have

$$0 < \langle \nu_G, e_3 \rangle \leq \langle \nu, e_3 \rangle \leq \sup_{\partial\Omega} \langle \nu, e_3 \rangle < 1 \tag{8.7}$$

along $\partial\Omega$. This proves the claim.

Having established the existence of G, there are three possibilities. First, there is a contact between M and G at an interior point p of both surfaces. Since $G \subset \overline{W}$ and $M \subset \partial W$, M and G are tangent at p and G lies above M. The tangency principle implies $M = G$, proving the result. A second possibility is that at a boundary point $p \in \partial\Omega$, $\langle \nu_G(p), e_3 \rangle = \langle \nu(p), e_3 \rangle$. Then (8.7) implies that both surfaces are tangent at p and G lies above M. Then the boundary version of the tangency principle gives $M = G$ again. The last case is that $int(G) \subset W$ and we have a strict inequality in (8.7) along $\partial\Omega$. The reasoning follows as in the proof of Theorem 5.2.1 using the flux formula, thereby obtaining a contradiction. □

8.2 The Bounded Slope Condition

A first example where the method of continuity applies is in the search for minimal graphs ($H = 0$). In such case, the solvability of (8.1) for arbitrary functions $\varphi \in C^0(\partial\Omega)$ is equivalent to the (non-strict) convexity of the domain Ω [Fin54, Fin65]. In this section we shall prove Finn's result assuming that Ω is strictly convex and φ is smooth on $\partial\Omega$. The basic feature is that, under these hypothesis, φ satisfies a bounded slope condition, which is defined as follows [GT01, Sects. 11.3, 12.4].

Definition 8.2.1 Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let φ be a function of $\partial\Omega$. Let $\Gamma = \{(q, \varphi(q)) \in \mathbb{R}^3 : q \in \partial\Omega\}$ denote the graph of φ. We say that Γ (or φ) satisfies a bounded slope condition with constant $K > 0$ if for each $p \in \Gamma$ there are two planes Π_p^+ and Π_p^- through the point p such that Γ lies between Π_p^+ and Π_p^- and the slopes of these planes are bounded by K.

We write $p = (q, \varphi(q))$. As the planes Π_p^+ and Π_p^- contain the point p, they are given by linear equations of the form

$$z = \pi_p^+(x, y) = \langle n_q^+, (x, y) - q \rangle + \varphi(q)$$
$$z = \pi_p^-(x, y) = \langle n_q^-, (x, y) - q \rangle + \varphi(q),$$

where $n_q^+, n_q^- \in \mathbb{R}^2$. The unit vectors orthogonal to Π_p^+ and Π_p^- are

$$\frac{(-n_q^+, 1)}{\sqrt{1 + |n_q^+|^2}}, \quad \frac{(-n_q^-, 1)}{\sqrt{1 + |n_q^-|^2}},$$

respectively. The constant K says that $|n_q^+|, |n_q^-| \leq K$ for $q \in \partial\Omega$ and the angle $\theta_q^\pm$ between $\Pi_p^\pm$ and the vector $(0, 0, 1)$ satisfies

$$\left|\cos \theta_q^\pm\right| = \frac{1}{\sqrt{1 + |n_q^\pm|^2}} \geq \frac{1}{\sqrt{1 + K^2}}.$$

This means that the planes Π_p^+ and Π_p^- are not arbitrary tilted for all $p \in \Gamma$. The other two conditions of the bounded slope condition property read as

$$\pi_p^-(q') \leq \varphi(q') \leq \pi_p^+(q')$$

for any $q' \in \partial\Omega$ and $p \in \Gamma$.

Some properties induced by the bounded slope condition are the following:

1. The function φ is continuous. Indeed, if $q \in \partial\Omega$ and $\{q_n\} \subset \partial\Omega \to q$, then $\pi_p^-(q_n) \leq \varphi(q_n) \leq \pi_p^+(q_n)$. Letting $n \to \infty$ and by the continuity of π_p^+ and π_p^-, we deduce $\lim \varphi(q_n) = \varphi(q)$.
2. If Γ does not lie in a plane and Γ satisfies a bounded slope condition, then Ω is convex. Fix $q \in \partial\Omega$. For each $q' \in \partial\Omega$,

$$0 \leq \pi_p^+(q') - \pi_p^-(q') = \langle n_q^+ - n_q^-, q' - q \rangle.$$

Since Γ does not lie in a plane, $n_q^+ - n_q^- \neq 0$. This shows that the straight line

$$\left\{(x, y) \in \mathbb{R}^2 : \langle n_q^+ - n_q^-, (x, y) - q \rangle = 0\right\}$$

is a supporting line at each point of $\partial\Omega$, proving the convexity of $\partial\Omega$.

The following result is due to P. Hartman [Har66].

Proposition 8.2.2 *Assume that $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain. If Ω is strictly convex and $\varphi \in C^\infty(\partial\Omega)$, then φ satisfies a bounded slope condition.*

We remark that the converse of this proposition holds in the following sense: if φ is not the restriction of a linear map (or equivalently, Γ is not a plane curve) and φ satisfies a bounded slope condition, then Ω is necessarily convex.

Given the definition of the bounded slope condition and Hartman's result, we can now prove the following existence result for minimal graphs.

Theorem 8.2.3 *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex bounded domain and $\varphi \in C^\infty(\partial\Omega)$. Then there exists a solution of $P(0,\varphi)$ which is smooth on $\overline{\Omega}$.*

Proof If Γ is a plane curve, the planar domain bounded by Γ is itself the solution of the Dirichlet problem. Suppose that Γ does not lie in a plane. Then Γ satisfies a bounded slope condition with constant $K > 0$. By the continuity method, consider the 1-parameter family of Dirichlet problems $P(0, t\varphi)$.

Claim *The curves $\Gamma_t = \{(q, t\varphi(q)) \in \mathbb{R}^3 : q \in \partial\Omega\}$ satisfy a bounded slope condition for the same constant K.*

Proof of the claim Given $p = (q, t\varphi(q))$, consider the planes $\Pi_p^{+,t}$ and $\Pi_p^{-,t}$ defined by

$$z = \pi_p^{+,t}(x, y) = \left\langle n_q^{+,t}, (x, y) - tq \right\rangle + t\varphi(q),$$
$$z = \pi_p^{-,t}(x, y) = \left\langle n_q^{-,t}, (x, y) - tq \right\rangle + t\varphi(q),$$

where $n_q^{+,t} = tn_q^+$ and $n_q^{-,t} = tn_q^-$. Then for each $q \in \partial\Omega$, $p = (q, t\varphi(q)) \in \Pi_p^{\pm,t}$ and for each $q' \in \partial\Omega$,

$$\pi_p^{-,t}(q') \le t\varphi(q') \le \pi_p^{+,t}(q').$$

Finally, $|n_q^{\pm,t}| = t|n_q^\pm| \le tK \le K$. □

In the scheme of the continuity method described in the preceding section (where now $H = 0$), the set $\mathscr{S}$ is open in $[0, 1]$ and $\mathscr{S} \neq \emptyset$. Let M_t denote the graph of the solution u_t of $P(0, t\varphi)$. Fix $p = (q, t\varphi(q)) \in \Gamma_t = \partial M_t$. Notice that Γ_t is not a plane curve because Γ is not. This implies that M_t is not a planar domain. We are going to estimate $|\nabla u_t|$ along $\partial\Omega$, which is sufficient by Lemma 8.1.11. For this purpose, consider the planes $\Pi_p^{+,t}$ and $\Pi_p^{-,t}$. Move $\Pi_p^{+,t}$ up vertically until it does not touch M_t. Next displace $\Pi_p^{+,t}$ down until it first touches M_t. Since $\Pi_p^{+,t}$ and M_t are minimal surfaces, the tangency principle asserts that this contact must occur when the plane $\Pi_p^{+,t}$ and the graph M_t touch in a common boundary point. Since $\Pi_p^{+,t}$ in its original position contains the point p and the bounded slope condition says that the boundary curve Γ_t lies below $\Pi_p^{+,t}$, the plane must return to its initial position.

Similarly, the surface M_t lies above $\Pi_p^{-,t}$ and both surfaces have a common point p. Furthermore, the tangency principle implies again that T_pM_t lies strictly between $\Pi_p^{-,t}$ and $\Pi_p^{+,t}$.

We apply Lemma 8.1.12 for the functions u_t, $\pi_p^{+,t}$ and $\pi_p^{-,t}$ obtaining

$$|\nabla u_t|(q) \leq \max\left\{\left|\nabla n_q^{-,t}\right|, \left|\nabla n_q^{+,t}\right|\right\} \leq tK \leq K.$$

This proves that we have a priori estimates of the gradient of the solutions of $P(0, t\varphi)$ showing the existence of a solution of $P(0, \varphi)$.

Furthermore, we have proved that the gradient $|\nabla u|$ is bounded up to $\partial\Omega$ and thus $|\nabla u|$ does not blow-up along $\partial\Omega$. This implies that the solution of $P(0, \varphi)$ is differentiable on $\overline{\Omega}$. □

Other recent results on the bounded slope condition in the context of the mean curvature equation appear in [AM09, FR03, Rip02].

8.3 Graphs on Convex Domains

In this section we study the Dirichlet problem when the boundary curve is planar, i.e. $\varphi = 0$. After a reflection about the plane P, we reduce the problem to the case $H > 0$. We will assume in addition that Ω is a convex domain. Some of the results in this section appear in [Lop99b, Lop01b, Lop02a, LM96, Rip01].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, which we assume to be strictly convex in the sense that the curvature κ of $\partial\Omega$ computed with the inward orientation is positive. Let

$$\kappa_0 = \min_{q \in \partial\Omega} \kappa(q) > 0.$$

Theorem 8.3.1 *Let $H > 0$. Consider a strictly convex bounded domain $\Omega \subset \mathbb{R}^2$. If $\kappa \geq H$, then there is a solution of $P(H, 0)$.*

Proof First we assume that $\kappa_0 > H$ and we will prove the existence of a solution a solution to $P(H, 0)$, which will be, in fact, smooth in $\overline{\Omega}$. For the existence of a solution to $P(H, 0)$, we shall provide two proofs. The first one is geometric using spherical caps as barriers to obtain the estimates of the slope of the graph along ∂M. Consider a solution u of (8.1)–(8.2) for $\varphi = 0$ and let M be the graph of u. There is no loss of generality in assuming $u > 0$ and so, the Gauss map of M points downwards. In view of Proposition 8.1.14, we need to control the slope of M along $\partial\Omega$. Let $q \in \partial\Omega$ and $r = 1/\kappa_0$. By the rolling property for strictly convex curves that appeared in the proof of Theorem 5.3.3, let C_r denote a circle of radius r such that $q \in C_r \cap \partial\Omega$ and $\Omega \subset D_r$, where $D_r \subset P$ is the domain bounded by C_r. Let $C_{1/H}$ be a circle of radius $1/H$ and concentric to C_r. Notice that $r < 1/H$.

Let us consider a hemisphere S_H of radius $1/H$ supported on P whose boundary is $C_{1/H}$ and such that $int(S_H)$ lies above P. Let us descend S_H until its intersection with P is C_r. Denote by S_r the piece of S_H above P at this position. The surface S_r is a small spherical cap and its mean curvature is $1/H$. See Fig. 8.1.

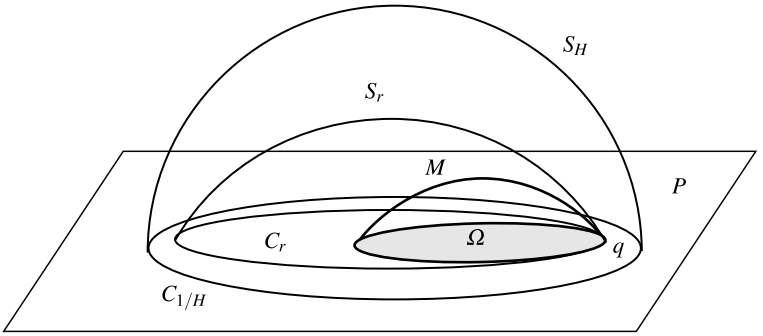

Fig. 8.1 Proof of Theorem 8.3.1. Along the boundary $\partial\Omega$, the slope of M is less than that of S_r

We prove now that the graph M lies in the domain $W \subset \mathbb{R}^3$ bounded by $S_r \cup D_r$. For this, we move S_r up by vertical translations until it no longer intersects M and next, move S_r down until it first intersects with M. Since the mean curvature of S_r is H and $\Omega \subset D_r$, the tangency principle implies that there is no contact before S_r arrives at its original position. In this position, the slope of M at q is less than that of S_r, which is a constant that only depends on r, and this yields the a priori estimates of $|\nabla u|$ along $\partial\Omega$. In fact, and since S_r is a smooth graph on $\overline{D_r}$ along C_r, we have $\sup_{\overline{\Omega}} |\nabla u| < \infty$.

The second proof is more analytic but it will be used in other contexts (see Theorems 11.2.5 and 11.3.1). Moreover, we will not obtain a priori estimates of $|\nabla u|$ along $\partial\Omega$ but directly in $\overline{\Omega}$. Denote by x the position vector of the surface M. From the proof of Theorem 2.3.5, we know that $\Delta(H\langle x, e_3\rangle + \langle N, e_3\rangle) \geq 0$ and by the maximum principle,

$$H\langle x, e_3\rangle + \langle N, e_3\rangle \leq \max_{\partial M}\big(H\langle x, e_3\rangle + \langle N, e_3\rangle\big).$$

Let $q \in \partial M = \partial\Omega$ be a point where the function $H\langle x, e_3\rangle + \langle N, e_3\rangle$ attains its maximum. As $\langle q, e_3\rangle = 0$, we have

$$H\langle x, e_3\rangle + \langle N, e_3\rangle \leq \big\langle N(q), e_3\big\rangle, \tag{8.8}$$

and for all $v \in T_q M$ pointing inside M, we find that $H\langle v, e_3\rangle + \langle dN_q(v), e_3\rangle \leq 0$. In particular, we choose $v = \nu(q) = \nu_q$, the inward unit conormal vector of M along $\partial\Omega$ at q. Then

$$H\langle \nu_q, e_3\rangle + \langle dN_q \nu_q, e_3\rangle \leq 0.$$

Since the boundary ∂M is a plane curve, we have

$$\big(H + \langle dN_q \nu_q, \nu_q\rangle\big)\langle \nu_q, e_3\rangle \leq 0.$$

As $\langle x, e_3\rangle \geq 0$ on Ω, it follows that $\langle \nu, e_3\rangle \geq 0$ on $\partial\Omega$ and the last inequality yields $H + \langle dN_q \nu_q, \nu_q\rangle \leq 0$. By the definition of the mean curvature H,

$$2H = -\langle dN_q t_q, t_q\rangle - \langle dN_q \nu_q, \nu_q\rangle,$$

where t_q is the unitary tangent vector along the boundary at q. Then

$$-H - \langle dN_q t_q, t_q\rangle = H + \langle dN_q v_q, v_q\rangle \leq 0.$$

Thus

$$H \geq -\langle dN_q t_q, t_q\rangle. \tag{8.9}$$

On the other hand, the normal curvature in the direction of t_q is

$$-\langle dN_q t_q, t_q\rangle = \big\langle N(q), \alpha''(q)\big\rangle = \kappa(q)\big\langle N(q), n(q)\big\rangle, \tag{8.10}$$

where $\alpha = \alpha(s)$ is an arc length parametrization of $\partial\Omega$, $\alpha'(0) = t_q$ and $n(q)$ is the unit normal vector to the boundary curve. The assumption $\kappa_0 > H$ and (8.9)–(8.10) gives

$$\big\langle N(q), n(q)\big\rangle \leq \frac{H}{\kappa(q)} \leq \frac{H}{\kappa_0} < 1.$$

In particular,

$$\langle v_q, e_3\rangle = \big\langle N(q), n(q)\big\rangle \leq \frac{H}{\kappa_0} < 1.$$

This proves that the slope of M is bounded by a constant, say, $H/\kappa_0 < 1$, depending only on the initial data. Hence it follows that there is a constant $C < 0$, depending only on H and Ω such that $\langle N(q), e_3\rangle \leq C$. From (8.8), we see that

$$\langle N, e_3\rangle \leq -H\langle x, e_3\rangle + C \leq C,$$

which gives an estimate of $|\nabla u|$ on $\overline{\Omega}$ according to (8.5).

So far, we have obtained the existence of a solution of $P(H, 0)$ for each H with $0 < H < \kappa_0$. Moreover, and since the gradient is bounded from above in $\overline{\Omega}$ depending only on the initial data, the solution obtained is smooth in $\overline{\Omega}$. Now, we proceed by proving the existence of a solution of $P(H, 0)$ in the case $H = \kappa_0$. The reader may consider the particular example where Ω is a round disk of radius $r = 1$ (and $\kappa_0 = 1$). Now the problem is to solve the Dirichlet problem for $H = 1$, where the solution is $u(x, y) = -\sqrt{1 - x^2 - y^2}$ which is not smooth on $x^2 + y^2 = 1$. Then we approximate by solving $H_n \to 1$ by small spherical caps.

Let us consider an increasing sequence $H_n \to H$. Let u_n be a solution of $P(H_n, 0)$ in Ω, which exists because $\kappa_0 > H_n$. By the monotonicity of H_n and the tangency principle, the sequence $\{u_n\}$ is monotonically increasing and converges uniformly on compact subsets of Ω. Let $u = \lim u_n$. Standard compactness results involving the Ascoli-Arzelá theorem guarantee that $u \in C^2(\Omega)$ and $Q_{-H}(u) = 0$ as in Proposition 8.1.7: because the orientation points downwards, in (8.1) we are using the operator Q_{-H}.

It remains to check that $u \in C^0(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. Let $p \in \partial\Omega$ and $\{p_m\} \subset \Omega$ with $p_m \to p$. Consider the hemisphere S_H as above and let D_R be the open disk of radius $R = 1/H$ such that $S_H = \text{graph}(v)$, with $v \in C^\infty(D_R) \cap C^0(\overline{D_R})$. Place

D_R such that $p \in \partial D_R$. We know that $\overline{\Omega} \subset D_R$ and by the tangency principle, $0 < u_n < v$ on Ω. For each $n \in \mathbb{N}$, $0 < u_n(p_m) < v(p_m)$. Then $0 \leq u(p_m) \leq v(p_m)$. Letting $m \to \infty$, $0 \leq u(p) \leq 0$. This proves the continuity of u up to $\partial\Omega$ and that $u = 0$ on $\partial\Omega$. □

In the proof of the preceding result we have used spherical caps as barrier surfaces. In the next result, we apply a similar argument but with pieces of cylinders, specifically with half cylinders and quarters of cylinders. By a quarter of a cylinder we mean the closure of each one of four connected components obtained by the intersection of an infinite cylinder of circular cross-section with two orthogonal planes containing its axis. It is worth noting that in the next theorem, the bounded domain Ω is merely convex and so $\partial\Omega$ might contain pieces of straight lines.

Theorem 8.3.2 *Let $H > 0$. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain included in an open strip of width $w > 0$. If $H \leq 1/w$, then there exists a solution of $P(H, 0)$.*

Proof As in the preceding theorem, we distinguish two cases, depending on whether $H < 1/w$ or $H = 1/w$. We first assume that $H < 1/w$. Let M be an H-graph on Ω with $\partial M = \partial\Omega$ and suppose that M lies above the plane P with equation $z = 0$.

Claim *There exists a number $\delta > 0$, $\delta = \delta(\Omega, H)$, such that the height h of M with respect to P satisfies*

$$h < \frac{1}{2H} - \delta. \tag{8.11}$$

Proof of the claim As $w < 1/H$, we place Ω between two parallel lines $L_1' \cup L_2'$ such that $\operatorname{dist}(L_1', L_2') < 1/H$. Let S_H be a half-cylinder of radius $1/(2H)$ such that $S_H \subset P^+ = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ and $\partial S_H \subset P$. Observe that ∂S_H is formed by two parallel lines $L_1 \cup L_2$ whose distance is $1/H$.

Consider the strip $D_H \subset P$ bounded by ∂S_H. Place S_H so that $L_1' \cup L_2' \subset D_H$ (and consequently, the four straight lines L_i' and L_i are parallel) and $\operatorname{dist}(L_1, L_1') = \operatorname{dist}(L_2, L_2')$ as depicted in Fig. 8.2, left. We assert that it is possible to lower S_H until its intersection with the plane P is $L_1' \cup L_2'$ and the surface M remains below S_H. Denote by ϕ_t the vertical translation $\phi_t(p) = p + te_3$, $t \in \mathbb{R}$. Move S_H up until it does not touch M, that is, consider $t > 0$ sufficiently large so that $\phi_t(S_H) \cap M = \emptyset$: see Fig. 8.2, right. Next we move S_H down. Taking into account that M and S_H have the same (constant) mean curvature with the downward orientation, a first contact point p for some $t > 0$ is impossible because in such case, M and S_H are tangent at p, $M \geq S_H$ around p and the tangency principle would yield a contradiction. Thus S_H returns to the initial position.

We proceed in the same way by moving S_H down slightly until the time $t_0 < 0$ such that $\phi_{t_0}(S_H) \cap P = L_1' \cup L_2'$. Let $S^H = \phi_{t_0}(S_H) \cap P^+$. Thus M lies below S^H, in particular, the height h of M is less than that of S^H. Now the height of S^H is $1/(2H) - \delta$, where δ depends only on $L_1' \cup L_2'$, i.e. on Ω and the cylinder S_H. We

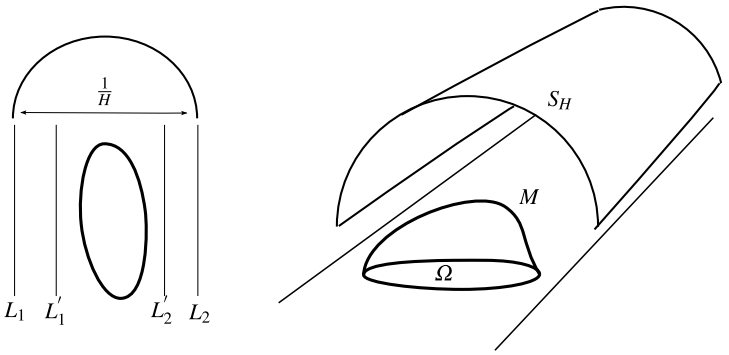

Fig. 8.2 Proof of Theorem 8.3.2. The domain Ω included in a strip determined by $L_1' \cup L_2'$. This strip defines a piece of a cylinder S_H that is a graph on the strip

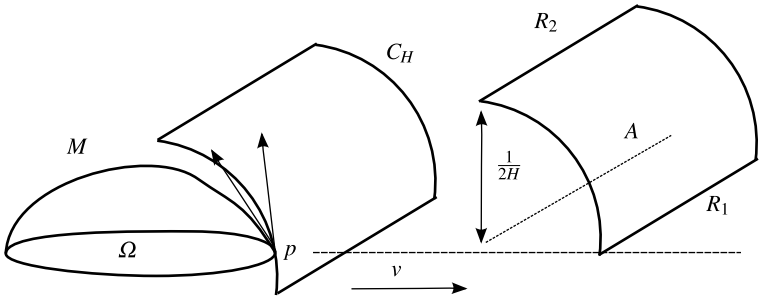

Fig. 8.3 Proof of Theorem 8.3.2. Along ∂M, the slope of M is less than that of the cylinder C_H with the same height as S

point out that we have not used that Ω is a convex domain but Ω lies in a strip of width $1/H$. □

Having proved the height estimate (8.11), we now show the existence of a priori estimates of the slope of M along the boundary. Let $p \in \partial M = \partial\Omega$. Let C_H denote a quarter of a cylinder of radius $1/(2H)$ whose boundary is formed by two parallel lines $\partial C_H = R_1 \cup R_2$. If A is the axis of revolution of C_H, we place C_H so that $C_H \subset P^+$, $R_1 \cup A \subset P$ and R_2 lies at height $1/(2H)$. The surface C_H separates the slab $\{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq 1/(2H)\}$ into two connected components one of which, C_H^+, is convex. Position C_H in such way that A is parallel to $T_p\partial\Omega$, displacing horizontally if necessary, assume that M is included in C_H^+. Descend C_H until the height of C_H is $1/(2H) - \delta/2$, where δ is the number given in (8.11). Denote by C_H again the portion of C_H above the plane P. In this position, C_H is a graph over a strip on the plane $z = 0$ whose boundary is given by two straight lines parallel to $T_p\partial\Omega$ and the mean curvature of C_H is H with the downward orientation. Displace C_H horizontally towards M in the opposite direction until C_H touches M: see Fig. 8.3. By the tangency principle, there is no interior contact point between

M and C_H. Hence we move C_H until its boundary touches ∂M exactly at the point $p \in \partial\Omega$ by the convexity of $\partial\Omega$. At p, the slope of M is less than that of C_H. On the other hand, the slope of C_H is a constant that depends only on the cylinder C_H and Ω. This shows the existence of an a priori estimate of the slope of M along its boundary, proving the result. Moreover, the solution u is smooth in $\overline{\Omega}$ since C_H is never vertical along its boundary.

It remains to prove the existence in the case $H = 1/w$. The argument is similar to that in the proof of Theorem 8.3.1. Consider an increasing sequence $H_n \to H$ and let u_n be the corresponding solutions of $P(H_n, 0)$ in Ω whose existence is guaranteed because $H_n < H = 1/w$. Comparing with the half cylinder S_H, we know that $0 < u_n < 1/(2H)$ in Ω for all n and we let $u = \lim u_n$. The function u satisfies $Q_{-H}(u) = 0$ in Ω and $u_n \to u$ uniformly in compact sets of Ω. In particular, $0 \leq u \leq 1/(2H)$ in Ω. We prove that u can be extended continuously to $\partial\Omega$. Let $p \in \partial\Omega$ and $\{p_m\} \subset \Omega$ with $p_m \to p$. Following the same process as above, we move the quarter cylinder C_H until ∂C_H touches $\partial\Omega$ at the point p. Let $v \in C^2(D_p) \cap C^0(\overline{D_p})$ be the function whose graph is C_H and where $D_p \subset \mathbb{R}^2$ is the open strip defined by $R_1 \cup A$. By the tangency principle, we have that C_H lies above u_n for all n around p. Then in a neighborhood of p in $\Omega \cap D_p$, $0 < u_n(p_m) < v(p_m)$, and taking limits, we finally obtain $u(p) = 0$. □

The next result holds again for non-strictly convex domains and it uses cylinders as in the preceding theorem.

Theorem 8.3.3 *Let $H > 0$. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and let L be the length of $\partial\Omega$. If*

$$H \leq \frac{\sqrt{3}\pi}{L}, \tag{8.12}$$

then there is a solution of $P(H, 0)$.

Proof First, we prove the existence for $H < \sqrt{3}\pi/L$, obtaining a smooth solution in $\overline{\Omega}$. If $H = \sqrt{3}\pi/L$, the argument is similar to that for Theorem 8.3.2 and we omit it. Let $t \in [0, 1]$ and consider the graphs M_t of the solutions of $P(tH, 0)$. Without loss of generality, suppose that M_t lies above the plane $z = 0$. Observe that Ω is simply connected since it is convex. Then M_t is a topological disk and we make use of the isoperimetric inequality (6.7) for each M_t:

$$L^2 - 4\pi A_t + t^2 H^2 A_t^2 \geq 0,$$

where A_t is the area of M_t. Using (8.12), the discriminant $16\pi^2 - 4L^2t^2H^2$ of the above quadratic inequality on t is positive because

$$16\pi^2 - 4L^2t^2H^2 \geq 16\pi^2 - 4L^2H^2 > 4\pi^2 > 0.$$

Then

$$A_t t^2 H^2 \leq 2\pi - \sqrt{4\pi^2 - L^2t^2H^2} \quad \text{or} \quad A_t t^2 H^2 \geq 2\pi + \sqrt{4\pi^2 - L^2t^2H^2}$$

for each $t \in [0, 1]$. By Corollary 8.1.6, we know the existence of H-graphs in some interval around $t = 0$. As A_t is a continuous map on t, and the first inequality holds for $t = 0$, this equality is valid for any $t \in [0, 1]$. In particular, for $t = 1$ we have

$$\frac{AH}{2\pi} \leq \frac{2\pi - \sqrt{4\pi^2 - L^2H^2}}{2\pi H}, \tag{8.13}$$

where $A = A_1$ is the area of $M = M_1$. Define $\delta = \delta(H, L)$ by

$$\delta = \frac{1}{2H} - \frac{2\pi - \sqrt{4\pi^2 - L^2H^2}}{2\pi H}.$$

The assumption $H < \sqrt{3}\pi/L$ and (8.13) imply that $0 < \delta < 1/(2H)$ and

$$\frac{AH}{2\pi} \leq \frac{1}{2H} - \delta.$$

The estimate (6.1) asserts that the height of M with respect to P is less than $1/2H - \delta$. Now we employ the same argument as in Theorem 8.3.2 to conclude the proof. □

The next result is due to L.E. Payne and G.A. Philippin and now it is a consequence of the preceding theorem [PP79].

Corollary 8.3.4 *Let $H > 0$. Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain such that the curvature κ of $\partial\Omega$ with respect to the inward orientation satisfies*

$$k \geq \frac{2H}{\sqrt{3}}.$$

Then there is a solution of $P(H, 0)$.

Proof By the assumption, the length L of $\partial\Omega$ is less than or equal to that of the circle of curvature $2|H|/\sqrt{3}$, namely,

$$L \leq 2\pi \frac{\sqrt{3}}{2H} = \frac{\sqrt{3}\pi}{H}.$$

Now we apply Theorem 8.3.3. □

In the next result, we assume a control of the area of Ω.

Theorem 8.3.5 *Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain. If $H > 0$ satisfies*

$$a(\Omega)H^2 \leq \frac{\pi}{2}, \tag{8.14}$$

then there is a solution of $P(H, 0)$.

Proof Inequalities (6.9) and (8.14) imply

$$h \leq \frac{a(\Omega)|H|}{2(\pi - a(\Omega)H^2)} \leq \frac{1}{2H}.$$

The reasoning is as in Theorem 8.3.2, distinguishing cases according to whether or not we have a strict inequality in (8.14). □

Note that the inequality (8.14) is not optimal. Indeed, if Ω is a round disk, there is a family of small spherical caps that are graphs on Ω with mean curvature H for each H with $0 \leq H^2 \leq \pi/a(\Omega)$. In view of this example, one is led to conjecture that it suffices to suppose that $a(\Omega)H^2 \leq \pi$, which is suitable in the case when Ω is a round disk.

On the other hand, the classical isoperimetric inequality in the plane states that $L^2 \geq 4\pi a(\Omega)$. However this inequality does not allow us to relate Theorem 8.3.3 and Theorem 8.3.5.

Remark 8.3.6 S. Montiel proved the solvability of $P(H, 0)$ assuming that $a(\Omega)H^2 < \alpha^2\pi$, where $\alpha = (\sqrt{5} - 1)/2$ is the golden ratio: $\alpha^2 \approx 0.3819$ [Mon01].

We call the reader's attention to the fact that in the context of the Plateau problem for parametric surfaces, there are results of the same flavor as Theorem 8.3.5. A theorem formulated by H.C. Wente [Wen69] and sharpened later by K. Steffen [Ste76] proves the existence of an H-disk immersed in $\mathbb{R}^3$ spanning a given Jordan curve Γ provided that $a_\Gamma H^2 < 2\pi/3$, where a_Γ is the least spanning area of Γ. If Γ is a plane curve, we have $a_\Gamma = a(\Omega)$, where Ω is the planar domain bounded by Γ.

We end this section by considering the case where the boundary curve in the Dirichlet problem is not planar. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $H \in \mathbb{R}$. Firstly we ask for those conditions on Ω and H such that the Dirichlet problem $P(H, \varphi)$ is solvable for *arbitrary* φ. This question was solved by J. Serrin in [Ser69b] (see also [Ser70] when Ω is not smooth), extending previous results of R. Finn for minimal graphs. Comparing with the case $\varphi = 0$ (Theorem 8.3.1), it is clear that the solvability for arbitrary boundary conditions must require stronger assumptions on the convexity of Ω.

Theorem 8.3.7 (Serrin) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $\kappa > 0$ be the curvature of $\partial\Omega$ with respect to the inward orientation. If $H \in \mathbb{R}$ satisfies*

$$\kappa \geq 2H > 0,$$

then for any $\varphi \in C^0(\partial\Omega)$ there exists a unique solution of $P(H, \varphi)$.

J. Serrin also showed that *the condition $\kappa \geq 2H > 0$ is necessary for the existence of solutions with arbitrary boundary values* in the sense that if the property $\kappa \geq 2H$

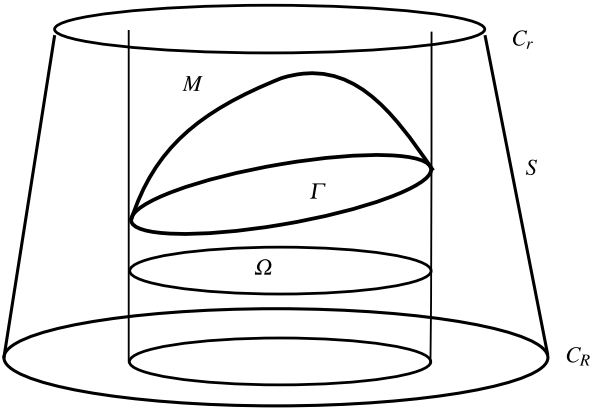

Fig. 8.4 Proof of Theorem 8.3.7. Along the boundary Γ, the slope of M is less than that of the truncated cone S

fails at a boundary point, it is possible to find a continuous function φ on $\partial\Omega$ such that the problem $P(H, \varphi)$ is not solvable.

Proof As usual, the proof distinguishes the cases $\kappa \geq 2H$ and $\kappa > 2H$. The methods for the case $\kappa \geq 2H$ are rather analytic and different. We refer to the original paper of Serrin [Ser69b] (see Remark 8.3.8 below). Here we only consider the case when $\kappa > 2H$ and φ is smooth. Let $z_0 = \min_{p\in\Gamma} z(p)$ and $z_1 = \max_{p\in\Gamma} z(p)$. If $z_0 = z_1$, φ is constant and we apply Theorem 8.3.1. Thus we suppose $z_0 < z_1$. We employ the continuity method as usual. In order to obtain a priori C^1-estimates, it suffices to study an H-graph M on Ω with boundary $\Gamma = \text{graph}(\varphi)$ (a similar argument works for u_t, $t \in [0, 1]$). We know from Theorem 2.3.5 the estimates

$$z_0 - \frac{1}{H} < u < z_1 + \frac{1}{H}, \tag{8.15}$$

with $M = \text{graph}(u)$. By Lemma 8.1.11, we shall control $|\nabla u|$ along $\partial\Omega$, and this we will do by means of truncated cones of appropriate mean curvatures and whose height is greater than that of M.

Let $h > 0$ satisfy $\kappa_0/2 > h > H$ and $\kappa_0 = \min_\Gamma \kappa$. Fix two positive numbers $r < R$ with

$$\frac{1}{\kappa_0} < r < R < \frac{1}{2h} < \frac{1}{2H}. \tag{8.16}$$

Let C_R and C_r denote two coaxial circles given respectively by

$$\alpha(s) = \left(R\cos(s), R\sin(s), z_0 - \frac{1}{H}\right), \qquad \beta(s) = \left(r\cos(s), r\sin(s), z_0 + \frac{1}{H}\right).$$

After a rigid motion, suppose that $\Omega \subset \{(x, y) : x^2 + y^2 < r^2\}$. Consider the truncated cone S defined by C_R and C_r parameterized by $\Psi(s, t) = t\alpha(s) + (1-t)\beta(s)$, $t \in [0, 1]$, $s \in \mathbb{R}$. See Fig. 8.4. We point out that S is a graph of a function v^+ defined on the annulus $D = \{(x, y) \in \mathbb{R}^2 : r^2 < x^2 + y^2 < R^2\}$. The height of S is

$m = (z_1 - z_0) + 2/H$ and the mean curvature H_S of S with the inward orientation is

$$H_S(s,t) = \frac{1}{2}\frac{m}{\sqrt{((R-r)^2 + m^2r^2)(Rt + r(1-t))^2}}.$$

This function is decreasing on t and so

$$H_S \geq H_S(s,1) = \frac{1}{2R}\frac{mr}{\sqrt{(R-r)^2 + m^2r^2}},$$

which is a constant. Since $1/(2R) > H$, take R sufficiently close to r satisfying (8.16) and

$$\frac{1}{2R}\frac{mr}{\sqrt{(R-r)^2 + m^2r^2}} > H.$$

In particular, $H_S > H$.

We now are in position to compare S with M. It is worth noting that any horizontal level curve at height z of S is a circle of radius ρ, $r \leq \rho \leq R$, and by the rolling property, this circle can be moved horizontally touching each point of the corresponding level curve of the cylinder $\partial\Omega \times \mathbb{R}$ at height z. Move S horizontally until it contacts M. Since Γ cannot touch C_R and C_r by the height estimate (8.15), the tangency principle guarantees that S touches M at a boundary point of M. Displace S horizontally touching each point of Γ (at different heights) which is possible by the convexity of Ω. Thus we have proved that given $q \in \partial\Omega$, there exists an annulus $D_q \subset \mathbb{R}^2$, a subdomain $\Omega_q^+ \subset \Omega$ with $\overline{\Omega_q^+} \subset D_q$, $q \in \partial\Omega_q^+ \cap \partial D_q$ and $\partial\Omega_q^+$ tangent to ∂D_q at q. Furthermore, $u \leq v^+$ on $\overline{\Omega_q^+}$ and $u(q) = v^+(q) = \varphi(q)$.

Now we use another truncated cone. Let S^* be the reflection of S about the horizontal plane $z = (z_0 + z_1)/2$. Then S^* is a truncated cone with boundary C_r in the plane $z = z_0 - 1/H$ and C_R in the plane $z = z_1 + 1/H$. If v^- is the function that defines the graph S^*, the same reasoning as above proves that given $q \in \partial\Omega$, $v^- \leq u$ in some subdomain Ω_q^- with $u(q) = v^-(q) = \varphi(q)$. Therefore $v^- \leq u \leq v^+$ on $\Omega_q^+ \cap \Omega_q^-$. Finally, Lemma 8.1.12 gives the control of the gradient of u at q:

$$|\nabla u|(q) \leq \left\{\left|\nabla v^+\right|(q), \left|\nabla v^-\right|(q)\right\} \leq \sup_D\left\{\left|\nabla v^+\right|, \left|\nabla v^-\right|\right\} = \sup_D\left|\nabla v^+\right|,$$

where the last equality holds by the symmetry of S^*. Observe that v^+ and v^- depend only on the cone S, i.e. on Ω and H, obtaining a priori estimates of $|\nabla u|$ along $\partial\Omega$. □

Remark 8.3.8 In the general case $\kappa \geq 2H$ in Theorem 8.3.7, the cylinder $\partial\Omega \times \mathbb{R}$ itself has mean curvature $\kappa/2$ and thus, it is greater than or equal to H. However $\partial\Omega \times \mathbb{R}$ is not a graph on Ω. Observe that if $r \to R$ in the above surface S, then S converges to a right cylinder with mean curvature $1/(2R) > H$. One expects that given $q \in \partial\Omega$, it is possible to slightly incline the cylinder in a neighborhood of the point $(q, \varphi(q)) \in \Gamma$, obtaining control of the slope of M around this point. The

proof given in [Ser69b] avoids these difficulties by studying the distance function to the boundary $\partial\Omega$.

The next result due to J. Ripoll [Rip02] treats non-zero boundary values again. In order to place the next theorem in the context of this section, recall that we have obtained a set of existence results of the Dirichlet problem for zero boundary data. If one wants that the boundary conditions to be arbitrary, then Serrin's result imposes stronger assumptions on the curvature κ of $\partial\Omega$. Between both types of results, we ask what assumptions should be assumed for a given function φ on $\partial\Omega$ in order to solve the Dirichlet problem. In [Rip02], sufficient conditions are expressed in terms of a boundary slope condition on φ.

Theorem 8.3.9 *Let $H > 0$. Let Ω be a bounded convex domain and let $\varphi \in C^\infty(\partial\Omega)$. Assume that φ satisfies a bounded slope condition with constant $K > 0$. Assume either one of the following assumptions*:

1. *The curvature κ of $\partial\Omega$ satisfies $\kappa_0 \geq H(1+K^2)^2$.*
2. *The domain Ω is included in a strip of width w such that $Hw \leq 1/(1+K^2)$.*
3. *The area of Ω satisfies $\sqrt{1+K^2}(2+K^2+K\sqrt{1+K^2})a(\Omega)H^2 < \pi$.*

Then there exists a solution of $P(H,\varphi)$.

Proof The proof in the three cases is similar, and we only present it for the first one. We assume that H satisfies $\kappa_0 > H(1+K^2)^2$. The case $\kappa_0 = H(1+K^2)^2$ is proved by taking an increasing sequence $H_n \to H$, obtaining a sequence of solutions u_n of $P(H_n,\varphi)$ and we then follow the proof of Theorem 8.3.1. Assume that the orientation points downwards. By the method of continuity, we have to give a priori estimates of $|\nabla u|$ along $\partial\Omega$ for a solution u of $P(H,\varphi)$ (Lemma 8.1.11). Let $M = \text{graph}(u)$.

Consider $p = (q,\varphi(q)) \in \partial M$, with $q \in \partial\Omega$. Let Π_p^+ be the plane given by the bounded slope condition. Let us intersect $\Omega \times \mathbb{R}$ with Π_p^+, obtaining a planar domain D_p. In this domain, we consider the Dirichlet problem $P(H,0)$. For the solvability, we apply Theorem 8.3.1, showing that there is a solution if the curvature of ∂D_p, namely κ_p, satisfies $\kappa_p \geq H$. We mention that for the cases (2) and (3), we use Theorems 8.3.2 and 8.3.5 respectively. We compute κ_p. If $\beta(s)$ is a parametrization of $\partial\Omega$ by the arclength, then

$$\alpha(s) = \beta(s) + \big(\langle \beta(s) - q, n_q^+ \rangle + \varphi(q)\big)e_3$$

parametrizes ∂D_p. A straightforward computation of κ_p gives

$$\kappa_p(s) = \kappa(s)\frac{|e_3 - n_q^+|}{(1+\langle \beta'(s), n_q^+\rangle^2)^{3/2}}.$$

As $e_3 \perp n_q^+$, then $|e_3 - n_q^+| > 1$. Using the hypothesis, we find that

$$\kappa_p(s) > \kappa(s)\frac{1}{(1+K^2)^{3/2}} \geq H\sqrt{1+K^2} > H. \tag{8.17}$$

This proves the existence of a solution $v_p \in C^2(\overline{D_p})$ of $P(H,0)$.

Now we prove that the graph G_p of v_p is, indeed, a graph on Ω. For this purpose, we know from Theorem 8.3.1 that the slope of G_p along its boundary is less than that of the H-spherical cap that gave the a priori estimates for the function v_p. This is expressed as

$$|\nabla v_p|^2 < \frac{H^2}{\kappa_{0p}^2 - H^2}$$

where $\kappa_{0p} = \min_{\partial D_p} \kappa_p$. But (8.17) yields

$$|\nabla v_p|^2 < \frac{H^2}{H^2(1+K^2) - H^2} \leq \frac{1}{K^2}.$$

Therefore $|\nabla v_p|$, which measures the slope of G_p with respect to n_q^+, satisfies

$$|\nabla v_p| \left| n_q^+ \right| < 1,$$

indicating that $G_p \subset \overline{\Omega} \times \mathbb{R}$. This ensures that there exists a $u_p \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $G_p = \text{graph}(u_p)$.

The graph G_p lies above the plane Π_p^+ and Π_p^+ lies above the boundary $\partial M = \text{graph}(\varphi)$ by the bounded slope condition. Using the tangency principle, we conclude that G_p lies above M, that is, $u_p \geq u$ in Ω and $u_p(p) = u(p)$. Let $\Pi_p^- = \text{graph}(\pi_p^-)$, which satisfies $p \in \Pi_p^- \cap M$. Since the orientation on M points downwards, the tangency principle implies that Π_p^- lies below M. Then

$$\pi_p^- \leq u \leq u_p \text{ in } \Omega \quad \text{and} \quad \pi_p^-(p) = u_p(p).$$

Lemma 8.1.12 asserts that $|\nabla u|(p) \leq \{|\nabla u_p|(p), K\}$. Since $|\nabla u_p|$ only depends on the initial data H, φ and Ω, we have obtained a priori estimates of $|\nabla u|$ along $\partial\Omega$, proving the result. □

Remark 8.3.10 Observe that if $\varphi = 0$ in Theorem 8.3.9, then we can take $K = 0$. Then the three assumptions correspond, respectively, with the results proved in Theorems 8.3.1, 8.3.2 and 8.3.5.

8.4 The Exterior Circle Condition

This section provides existence results for cmc graphs when the domain is not convex. While in the preceding section we used pieces of spheres and cylinders to solve the Dirichlet problem, here we show the role that nodoids play as barriers to get the desired C^1-estimates.

The key property of nodoids that we shall use is the following. Fix $H > 0$. A nodoid with mean curvature H contains pieces of catenoid-shape with mean curvature vector pointing outward. Given an H-surface, we move these pieces of

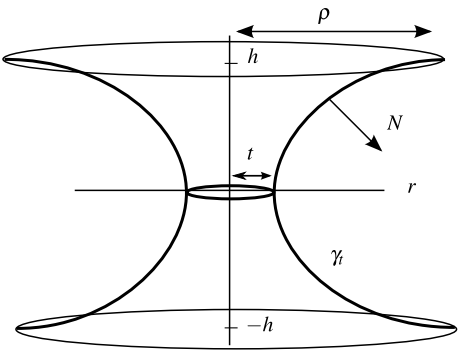

Fig. 8.5 The 1-parameter family of nodoids $\mathcal{N}_t$

H-nodoids from infinity towards to the surface with the correct orientation on the nodoid for comparison using the tangency principle. See Fig. 8.5.

Recall the construction of nodoids and some of their properties. Let $\gamma(u) = (r(u), 0, u)$ be a curve defined on an open interval I of real numbers, $0 \in I$, and $r > 0$. The surface of revolution rotating γ with respect to the z-axis is

$$X(u, \theta) = \big(r(u)\cos\theta, r(u)\sin\theta, u\big).$$

Choose the outward orientation $N(u, \theta) = (\cos\theta, \sin\theta, -r')/\sqrt{1 + r'^2}$ and assume that the mean curvature H is constant. The mean curvature is

$$2H = -\frac{1}{r(1 + r'^2)^{1/2}} + \frac{r''}{(1 + r'^2)^{3/2}}, \tag{8.18}$$

where the principal curvatures are each of the two summands on the right-hand side of (8.18). Multiplying (8.18) by rr', we derive a first integral

$$Hr^2 = -\frac{r}{\sqrt{1 + r'^2}} + c, \tag{8.19}$$

for some constant $c \in \mathbb{R}$. Depending on the initial conditions on (8.19), we obtain the different Delaunay surfaces [Del41]. So:

1. If $H = 0$, $r(u) = c\cosh(u/c)$, $c \neq 0$ and the surface is a catenoid.
2. If r is a constant function, the surface is a cylinder.
3. If $c = 0$, r describes a half circle $r(u) = \sqrt{1/H^2 - u^2}$ and the surface is a sphere.
4. In the other cases, we obtain unduloids and nodoids.

We focus on nodoids. There exist two numbers $h, \rho > 0$, where $r : [-h, h] \to [t, \rho]$ is an even function and the initial condition $r(0) = t > 0$ is the minimum value of r. Moreover, $r(h) = \rho$ and $r'(\pm h) = \pm\infty$. In particular, $H\rho^2 = c$. Since $r'(0) = 0$, from (8.19) we deduce $Ht^2 + t = c$, i.e.

$$t = \frac{-1 + \sqrt{1 + 4Hc}}{2H}. \tag{8.20}$$

Therefore $H > 0$ and so, $c > 0$. Moreover,

$$\rho = \sqrt{\frac{Ht^2 + t}{H}} = \sqrt{\frac{c}{H}}. \tag{8.21}$$

When convenient, we indicate the dependence of the parameter t by a subscript t and by (H) the dependence of H. Let $\mathcal{N}_t$ denote the nodoid produced by rotating γ_t when γ_t is defined on the interval $[-h, h]$. Notice that $\rho_t(H) - t(H)$ is increasing on t and

$$\rho_t(H) - t(H) = \frac{1}{2H}\left(1 - \frac{1}{2\sqrt{Hc} + \sqrt{1 + 4Hc}}\right) \to \frac{1}{2H}$$

as $t \to \infty$.

We now change coordinates. Let us view γ_t as a graph on the x-axis and express the nodoids as $X(x, \theta) = (x\cos\theta, x\sin\theta, z(x))$. The mean curvature H satisfies

$$Hx^2 = \frac{xz'}{\sqrt{1 + z'^2}} + c$$

and for the value $x = t$, we have

$$z(t) = 0, \qquad z'(t) = +\infty.$$

Therefore

$$h_t(H) = z(\rho) - z(t) = \int_t^\rho z'(x)dx = \int_t^\rho \frac{c - Hx^2}{\sqrt{x^2 - (c - Hx^2)^2}}dx$$
$$= \int_t^\rho \frac{H(\rho^2 - x^2)}{\sqrt{x^2 - H^2(\rho^2 - x^2)^2}}dx.$$

We collect all these properties below (see Fig. 8.5):

Proposition 8.4.1 *Let $H > 0$. There exists a 1-parameter family of nodoids $\{\mathcal{N}_t : t > 0\}$ with constant mean curvature H given by the rotation of a curve γ_t around the z-axis and with the following properties*:

(a) *The parameter t measures the radius of the waist circle C_t of $\mathcal{N}_t$, which is contained in the plane $z = 0$.*
(b) *The curve γ_t is a graph on the interval $(-h_t, h_t)$ and symmetric with respect to the r-axis.*
(c) *The curve γ_t has horizontal tangent at the heights $z = \pm h_t$. In particular, $\mathcal{N}_t$ is tangent to the slab $S_t = \{(x, y, z) \in \mathbb{R}^3 : |z| \le h_t\}$.*
(d) *The mean curvature vector points outside the bounded domain W_t determined by $\mathcal{N}_t$ and the slab S_t.*

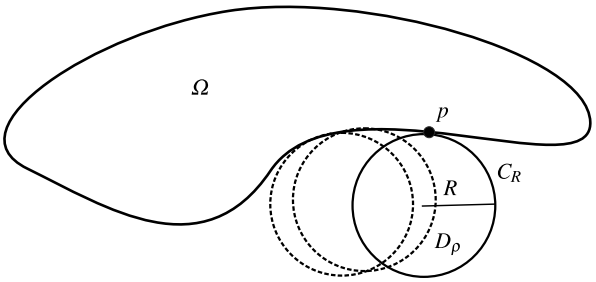

Fig. 8.6 The uniform exterior R-circle condition and the rolling property of C_R along $\partial\Omega$

(e) *The function $h_t(H)$ is strictly increasing on t and*

$$\lim_{t\to 0} h_t(H) = 0, \qquad \lim_{t\to\infty} h_t(H) = \frac{1}{2H}.$$

(f) *The function $\rho_t(H)$ is strictly increasing on t and*

$$\lim_{t\to 0} \rho_t(H) = 0, \qquad \lim_{t\to\infty} \rho_t(H) = \infty, \qquad \lim_{t\to\infty} \big(\rho_t(H) - t(H)\big) = \frac{1}{2H}.$$

(g) *The function $h_t(H)$ is decreasing on H.*

In this section, we shall prove the existence of a cmc graph provided that we can compare with pieces of nodoids: see similar settings in [Fin84, Fin88] on capillary problems. First, we need the following definition (see Fig. 8.6):

Definition 8.4.2 Let $R > 0$. A domain $\Omega \subset \mathbb{R}^2$ satisfies a uniform exterior R-circle condition if at each point $p \in \partial\Omega$, there exists a closed disk $D_p \subset \mathbb{R}^2 \setminus \Omega$ depending on p of radius R such that $D_p \cap \overline{\Omega} = \{p\}$.

If Ω is smooth, which it is the case that we are considering for solving the Dirichlet problem, the uniform exterior condition is equivalent to the condition that a circle C_R of radius R can roll outside Ω along $\partial\Omega$ touching each point of $\partial\Omega$ along this displacement. If Ω satisfies the uniform exterior R-circle condition for any $R > 0$, then Ω is convex, or in other words, a convex domain may be viewed as a domain that satisfies the property for $R = +\infty$.

Theorem 8.4.3 *Let $H > 0$ and let $\Omega \subset \mathbb{R}^2$ be a bounded domain that satisfies a uniform exterior R-circle condition. Let $h_R = h_R(H)$ be the number given in Proposition 8.4.1. Under any of the following assumptions, there exists a solution of $P(H, 0)$:*

1. *The domain Ω is included in a circular disk of radius $r = \sqrt{(2 - Hh_R)h_R/H}$.*
2. *The domain Ω is included in a strip of width less than $2h_R(H)$.*

3. *The length of $\partial\Omega$ satisfies*

$$L \leq 2\pi\sqrt{(2 - Hh_R)\frac{h_R}{H}}.$$

4. *The area of Ω satisfies*

$$a(\Omega) \leq \frac{2\pi h_R}{H(1 + 2Hh_R)}.$$

Proof As in Theorem 8.3.1, the proof distinguishes the cases that the solution is smooth in $\overline{\Omega}$ or only in Ω. In the first case, we are assuming in hypotheses (1) and (2) that the inclusions refer to the closure of Ω, whereas in assumptions (3) and (4), we impose strict inequalities on L and $a(\Omega)$.

We begin with the first case. We claim that any of the four assumptions implies that there exists a number h_R^*, $0 < h_R^* < h_R$, depending only on R, H and Ω such that the height of a solution u of $P(H, 0)$ is strictly less than h_R^*. Let P be the plane $z = 0$. Without loss of generality, assume $u \geq 0$ on Ω and denote by M the graph of u. Recall that the orientation of M points downwards. We distinguish the four cases.

1. Let us consider a hemisphere S of radius $1/H$ in the half-space $\{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ whose circular boundary is contained in P and such that the center of the circle ∂S agrees with that of the circular disk that contains Ω. Notice that the assumption on r implies $1/H > r$. Translate S upward so S is disjoint from M. By the tangency principle, translate S downward until ∂S arrives at P. We continue moving down until S touches $\partial\Omega$. When this occurs, a calculation proves that the height h_R^* of the part of S over P is strictly less than h_R.
2. Assume that the strip is $S = \{(x, y) \in \mathbb{R}^2 : -h_R < y < h_R\}$. As $\overline{\Omega} \subset S$, there exists a positive number w, $0 < w < h_R$, such that the straight lines on P given by $y = \pm w$, $z = 0$ define a strip $B_w \subset S$ with the property $\overline{\Omega} \subset B_w$. Let $(x_0, y_0) \in \Omega$. Let $\mathscr{N}_R$ be the H-nodoid of Proposition 8.4.1 for the parameter $t = R$. Position $\mathscr{N}_R$ so that its axis of revolution is parallel to the y-axis and the two coaxial circles of $\mathscr{N}_R$ project on ∂S. Moreover, we position $\mathscr{N}_R$ so that the projection of the lowest points of $\mathscr{N}_R$ are the points $(x_0, -h_R)$ and (x_0, h_R). Since M is compact, we lift up $\mathscr{N}_R$ until it does not touch M. Next we move it down until the first contact point. By the tangency principle, the contact cannot occur at interior points because the Gauss map of $\mathscr{N}_R$ points downwards. As $\partial\mathscr{N}_R$ projects on ∂S, displace $\mathscr{N}_R$ until $\partial\mathscr{N}_R$ touches the plane $z = 0$. In fact, we move down a bit more until $\mathscr{N}_R$ intersects ∂B_w. Denote by $\mathscr{N}_{R,w}$ the part of $\mathscr{N}_R$ that lies above the plane and such that $\partial\mathscr{N}_{R,w}$ touches ∂B_w at exactly two points. This proves that $u(x_0, y_0)$ is less than the height of $\mathscr{N}_{R,w}$, say, h_R^*.
3. In the same way as in Theorem 8.3.3, the area A of M satisfies $AH/(2\pi) < h_R$. This inequality, in connection with the height estimate (6.1), implies $u < h_R^*$ for some $h_R^* < h_R$.
4. The number h_R^* is obtained by the height estimate (6.9) and the assumption on $a(\Omega)$.

Assuming the height is less than h_R^*, we study the existence of such graph. Consider the initial nodoids $\mathscr{N}_t$ of constant mean curvature H defined in Proposition 8.4.1. Fix δ with $h_R^* < \delta < h_R$. As h_t is increasing on t, consider $r(\delta) < R$ such that the nodoid $\mathscr{N}_{r(\delta)}$ has height $h_{r(\delta)} = \delta$. Place $\mathscr{N}_{r(\delta)}$ in such a way that the waist circle $C_{r(\delta)}$ is contained in the plane P. We cut $\mathscr{N}_{r(\delta)}$ by a horizontal plane $z = \varepsilon > 0$, $\varepsilon = \varepsilon(\delta)$, close to P such that the part of $\mathscr{N}_{r(\delta)}$ over $z = \varepsilon$ has a height greater than or equal to h_R^* and the circle $\Gamma_\delta = \partial\mathscr{N}_{r(\delta)} \cap \{z = \varepsilon\}$ has a radius $R(\delta)$ with $R(\delta) < R$. Denote by $\mathscr{N}^R$ the part of $\mathscr{N}_{r(\delta)}$ over the plane $z = \varepsilon$. This surface is a graph on the annulus $\Omega_\delta = \{(x, y) \in \mathbb{R}^2 : R(\delta) < \sqrt{x^2 + y^2} < \rho(\delta)\}$, where $\rho(\delta)$ is the radius of the circle of $\mathscr{N}_{r(\delta)}$ at height δ. Move $\mathscr{N}^R$ vertically until Γ_δ meets P and let $D_\delta \subset P$ be round disk bounded by Γ_δ.

Given $p \in \partial\Omega$, position $\mathscr{N}^R$ so that $\Gamma_\delta \cap \overline{\Omega} = \{p\}$, which is possible by the uniform exterior R-circle condition and because $R(\delta) < R$. Displace $\mathscr{N}^R$ upward until it does not touch M and then let it descend. Since the height of M is less than $\mathscr{N}^R$, the tangency principle guarantees that the first intersection point between $\mathscr{N}^R$ and M occurs when $\mathscr{N}^R$ arrives at the plane P exactly at the point p. This proves that the slope of M at p does not exceed that of $\mathscr{N}^R$ at this position. This gives a uniform bound for $|\nabla u|$ along $\partial\Omega$ in terms of $\mathscr{N}^R$ and this completes the proof of the existence of a solution u. In addition, as $|\nabla u|$ is uniformly bounded along $\partial\Omega$, we have $u \in C^\infty(\overline{\Omega})$.

The second part of the proof makes use of the assumptions (1) and (2) given in the statement of the theorem and the equalities in hypotheses (3) and (4). Given $H \in \mathbb{R}$, consider an increasing sequence $H_n \to H$. For each $n \in \mathbb{N}$, let $u_n \in C^\infty(\overline{\Omega})$ be a solution of $P(H_n, 0)$. We remark that for each $H_n < H$, we can apply the existence result of the first part: case-by-case, we have:

1. It suffices to notice that

$$\frac{2h_R(H)}{H} - h_R(H)^2 < \frac{2h_R(H_n)}{H_n} - h_R(H_n)^2. \tag{8.22}$$

   The function $2a/x - a^2$, $a > 0$, is decreasing on x for $x > 0$. Then

$$\frac{2h_R(H)}{H} - h_R(H)^2 < \frac{2h_R(H)}{H_n} - h_R(H)^2.$$

   Next, the function $2x/a - x^2$ is increasing on $x \in (0, 1/a)$. For values H_n close to H, $h_R(H), h_R(H_n) < 1/H_n$. As $h_R(H) < h_R(H_n)$ by Proposition 8.4.1, (g), we conclude (8.22).
2. Fixing R, the function $h_R(H)$ is decreasing on H. Thus Ω is included in a strip of width $2h_R(H_n)$, and we apply the existence result.
3. From part (1), L satisfies

$$L < 2\pi\sqrt{\big(2 - H_n h_R(H_n)\big)\frac{h_R(H_n)}{H_n}}.$$

4. It suffices to prove

$$a(\Omega) < \frac{2\pi h_R(H_n)}{H_n(1+2H_n h_R(H_n))}. \tag{8.23}$$

As $H_n < H$, we have

$$\frac{2\pi h_R(H)}{H(1+2Hh_R(H))} < \frac{2\pi h_R(H)}{H_n(1+2H_n h_R(H))}.$$

On the other hand, the function $x/(1+ax)$, $a>0$, is increasing on x. As $h_R(H) < h_R(H_n)$, we obtain (8.23).

Again, compactness arguments prove the existence of $u \in C^2(\Omega)$ so that $u_n \to u$ monotonically and $Q_{-H}(u) = 0$. The proof that u is continuous on $\partial\Omega$ is similar to that of Theorem 8.3.1. Given $p \in \partial\Omega$, consider the piece of nodoid $\mathscr{N}^R$ of mean curvature H obtained when we position $\mathscr{N}_R$ in such way that $C_R \subset P$ and $\mathscr{N}^R$ is the part of $\mathscr{N}_R$ over the plane P. Let $v \in C^2(\Omega_R) \cap C^0(\overline{\Omega_R})$ be the function on the annulus $\Omega_R = \{(x,y) \in \mathbb{R}^2 : R < \sqrt{x^2+y^2} < \rho_R(H)\}$ whose graph is $\mathscr{N}^R$ and position $\mathscr{N}^R$ so that $C_R \cap \overline{\Omega} = \{p\}$. Let $D_R = \Omega_R \cap \Omega$ and consider the restriction of v to D_R. Now the argument is standard, taking into account that $0 < u_n < v$ in D_R. □

Theorem 8.4.3 may be viewed as an extension of previous results proved for convex domains. A convex domain $\Omega \subset \mathbb{R}^2$ satisfies a uniform exterior R-circle condition for all $R > 0$. Assume that $R = \infty$ and so, $h_\infty = \lim h_R(H) = 1/(2H)$ by Proposition 8.4.1, (e). Then the statements of Theorem 8.4.3 have been previously proved, indeed:

1. The domain Ω is included in a circular disk of radius $\sqrt{3}/(2H)$ (Corollary 8.3.4).
2. The domain Ω is included in a strip of width $1/H$ (Theorem 8.3.2).
3. The length L satisfies $LH \le \sqrt{3}\pi$ (Theorem 8.3.3).
4. The area $a(\Omega)$ satisfies $a(\Omega)H^2 \le \pi/2$ (Theorem 8.3.5).

Other existence results for cmc graphs on domains that satisfy a uniform exterior circle condition appear in [Bgn09, Rip01, RS08].

8.5 Radial Graphs in Euclidean Space

We end the chapter by considering the Dirichlet problem for the constant mean curvature equation on a domain of the unit sphere $\mathbb{S}^2$. The solutions are radial graphs according the next definition.

Definition 8.5.1 Let $\Omega \subset \mathbb{S}^2$ be a domain and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. The radial graph of u is the surface

$$M = \left\{x(q) = e^{u(q)}q : q \in \overline{\Omega}\right\},$$

where x denotes the position vector of M in $\mathbb{R}^3$.

From the geometric viewpoint, a radial graph M on $\mathbb{S}^2$ is characterized as a star-shaped surface relative to the center of $\mathbb{S}^2$ (in this case, the origin of $\mathbb{R}^3$), that is, each ray emanating from the origin intersects M transversally. Consequently, if N is a unit normal vector field on M, the support function $\langle N, x\rangle$ is either positive or negative on M. Throughout this section, we shall assume the orientation N is opposite to the radius vector, which is expressed as $\langle N, x\rangle < 0$ on M.

We shall treat the question of existence of radial graphs with constant mean curvature $H \in \mathbb{R}$ (briefly radial H-graphs). The pioneering work on this subject is due to T. Radó who showed that if a simple closed curve $\Gamma \subset \mathbb{R}^3$ has a single valued radial projection onto a plane convex curve, then Γ is the boundary of a minimal radial graph ([Rad32, Rad51]; a generalization in arbitrary dimensions appears in [Tau81]).

We establish the associated Dirichlet problem. Consider a radial graph $M = \text{graph}(u)$ on a domain $\Omega \subset \mathbb{S}^2$. Let e_1, e_2 be a local frame field on $\mathbb{S}^2$ and let ∇ be the covariant differentiation on $\mathbb{S}^2$. Let $\theta_{ij} = \langle e_i, e_j\rangle$ be the metric on $\mathbb{S}^2$. The metric g_{ij} on M is given by

$$g_{ij} = \langle \nabla_{e_i} x, \nabla_{e_j} x\rangle = e^{2u}(\theta_{ij} + \nabla_{e_i} u \nabla_{e_j} u),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^3$. The inward unit normal to x is

$$N\big(x(q)\big) = \frac{\nabla u(q) - q}{\sqrt{1 + |\nabla u|^2(q)}}, \quad q \in \Omega. \tag{8.24}$$

This map N satisfies the choice of orientation on a radial graph because $\langle N, x\rangle < 0$. The second fundamental form of x is

$$\sigma_{ij} = \langle \nabla_{e_i} \nabla_{e_j} x, N\rangle = \frac{e^u(\theta_{ij} + 2\nabla_{e_i} u \nabla_{e_j} u - \nabla_{e_i}\nabla_{e_j} u)}{\sqrt{1 + |\nabla u|^2}}.$$

After a straightforward computation, the mean curvature H satisfies the equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 2\left(-He^u + \frac{1}{\sqrt{1 + |\nabla u|^2}}\right), \tag{8.25}$$

where div denotes the divergence operator with the standard metric of $\mathbb{S}^2$. In this section, we study the Dirichlet problem when the boundary curve Γ is contained in $\mathbb{S}^2$, i.e. $\Gamma = \partial\Omega$. Then the boundary condition is written in terms of u as

$$u = 0 \quad \text{on } \partial\Omega. \tag{8.26}$$

The problem of existence of radial graphs of prescribed constant mean curvature presents some differences with the Dirichlet problem for a planar domain of $\mathbb{R}^2$, which we distinguish in this section with the term 'vertical graph'. This is the case for uniqueness, as shown the next example.

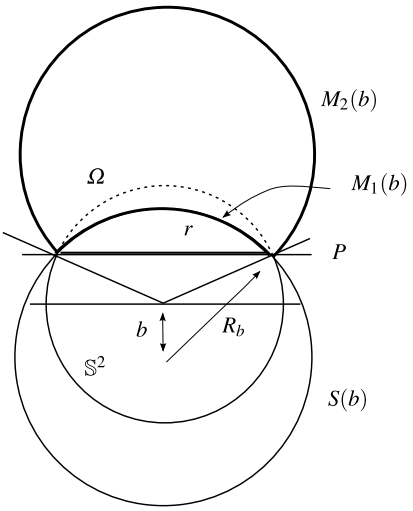

Fig. 8.7 Non-uniqueness of radial graphs for $H > 0$ on a domain $\Omega \subset \mathbb{S}^2$. Here Ω is represented by the *dashed line*

Example 8.5.2 Let

$$\frac{1}{\sqrt{2}} \leq r < 1, \qquad b \in \left[0, \frac{2r^2-1}{\sqrt{1-r^2}}\right], \qquad R_b = \sqrt{1+b^2+2b\sqrt{1-r^2}}.$$

Consider

$$S(b) = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z+b)^2 = R_b^2\right\}.$$

This surface $S(b)$ is a sphere centered at $(0, 0, -b)$ with radius R_b. Let $M_1(b)$ be the part of $S(b)$ above the plane P with equation $z = \sqrt{1-r^2}$ and let $M_2(b)$ be the reflection of $S(b)\backslash M_1(b)$ with respect to P as depicted in Fig. 8.7. The condition on b guarantees that both surfaces are radial graphs in the domain

$$\Omega = \left\{(x, y, z) \in \mathbb{S}^2 : z > \sqrt{1-r^2}\right\},$$

and have the same mean curvature $H_b = 1/R_b$. Furthermore, H_b satisfies $0 < H_b \leq 1$. Let us observe that the case $b = 0$ leads to $S(0) = \mathbb{S}^2$, hence $M_1(0) = \overline{\Omega}$ and $M_2(0)$ is the reflection of $\mathbb{S}^2\backslash\Omega$ with respect to the plane P.

However the uniqueness of the solution holds if $H \leq 0$.

Proposition 8.5.3 *Let M_1 and M_2 be two radial graphs on $\Omega \subset \mathbb{S}^2$ of two functions u_1 and u_2, respectively, and represent by H_1 and H_2 their mean curvature functions. If $u_1 \leq u_2$ on $\partial\Omega$ and $H_1 \leq H_2 \leq 0$ on Ω, then $u_1 \leq u_2$ in Ω. In particular, there is uniqueness of solutions of the Dirichlet problem (8.25)–(8.26) if $H \leq 0$.*

Proof Assume that $u_1 > u_2$ at a point of Ω. By homotheties $\phi_t(p) = tp$, with $t > 1$, we lift M_2 outwards until it lies entirely above the surface M_1 with respect to the

radius vector. Then lower it by homotheties again ($t \searrow 1$) until the initial contact point p with M_1 at $t = t_0 > 1$. Since the mean curvature is non-positive, homotheties with $t > 0$ increase the value of the mean curvature as we increase t. Let $M_2' = \phi_{t_0}(M_2)$. Since M_2' is tangent to M_1 at p, the normal vectors to M_2' and M_1 coincide. Then M_1 lies locally above M_2' around p and the comparison principle implies $H_2'(p) \leq H_1(p)$. We distinguish two cases:

1. Case $H_2(p) < 0$. Since $t_0 > 1$, we have

$$H_2'(p) = \frac{1}{t_0} H_2(p) > H_2(p) \geq H_1(p),$$

   obtaining a contradiction.
2. Case $H_2(p) = 0$. Here $H_2'(p) = 0$, hence $H_1(p) = 0$. The tangency principle implies that M_2' agrees with M_1 on an open subset around p. In this open subset, $H_2' = H_2/t_0 = H_1$. Since $H_1 \leq H_2$ and $t_0 > 1$, this implies $H_1 = H_2 = 0$ in this open set. With the same argument by using the tangency principle, and by connectedness, we conclude that $M_1 = M_2'$: a contradiction because $\partial M_1 \neq \partial M_2' = (1/t_0)\partial M_2$.

□

We know that if $H \neq 0$, a vertical H-graph with boundary in a plane P lies on one side of P. In our context, we ask if a radial H-graph on $\Omega \subset \mathbb{S}^2$ and with boundary $\partial\Omega$ lies on one side of Ω. In fact, the sign of H plays a fundamental role for the existence and uniqueness of radial H-graphs (recall that for zero boundary data, the sign of H is irrelevant for vertical graphs). More precisely, throughout this section, we will see that there are three intervals in the range of H where the behavior of the Dirichlet problem for radial graphs presents important differences, namely, $(-\infty, 0)$, $[0, 1]$ and $(1, +\infty)$. Definitively, this shows that the Dirichlet problem for radial H-graphs has a richness that well deserves its own interest. For existence and uniqueness of radial graphs, see also [AF08, AFR06, CG13, FR03, Swa04, Swa05].

One cannot expect solvability of the Dirichlet problem (8.25)–(8.26) without any assumptions on H and the size of the domain Ω. In fact, we have:

Proposition 8.5.4 *Consider a domain $\Omega \subset \mathbb{S}^2$ containing the closure of a hemisphere of $\mathbb{S}^2$. Then the Dirichlet problem (8.25) is not solvable for $H \leq 0$.*

Proof There is no loss of generality in assuming that Ω contains the upper hemisphere of $\mathbb{S}^2$. Then $\partial\Omega$ is included in the half-space $\mathbb{R}^3_- = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}$ and the boundary ∂M of a radial graph M on Ω lies in $\mathbb{R}^3_-$. Consider the horizontal plane P_t with equation $z = t$. Decrease t from $t = +\infty$ until P_t first touches M at $t = t_0$. As M is the radial graph of a positive function, the half-line $L = \{(0, 0, t) : t > 0\}$ intersects M. This means that $t_0 > 0$. If $p \in M \cap P_{t_0}$, $p = (p_1, p_2, t_0)$, the unit normal vector is $N(p) = (0, 0, -1)$ because $\langle N(p), p\rangle = -t_0 < 0$. Then $M \geq P$ around p and the comparison principle yields $H(p) \geq 0$. As $H \leq 0$, then $H = 0$

on M. The tangency principle implies that M is a domain included in the plane P_{t_0}. This contradicts the fact that ∂M lies in $\mathbb{R}^3_-$. □

In the simplest case where Ω is a round disk of $\mathbb{S}^2$, appropriate spherical caps with boundary $\partial\Omega$ are radial H-graphs on Ω. In general, the plane P containing the circle does not include the origin of $\mathbb{R}^3$. If Ω is included in a hemisphere and M is a radial graph, then $M \cap (P \setminus \overline{\Omega}) = \emptyset$ and Corollary 4.1.4 implies that the surface is a planar disk or a spherical cap. If Ω contains the closure of a hemisphere of $\mathbb{S}^2$, we do not know if umbilical surfaces are the only examples. In fact, we have the following

Conjecture *Planar disks and spherical caps are the only star-shaped surfaces in $\mathbb{R}^3$ with constant mean curvature and circular boundary.*

If the conjecture is true, it would be the boundary version of Jellet's theorem. On the other hand, J. Serrin studied the existence and uniqueness of radial graphs with prescribed mean curvature [Ser69b, §23]. Taking into account the choice of the orientation on a radial graph, he proved:

Theorem 8.5.5 *Let $H < 0$ and let $\Omega \subset \mathbb{S}^2$ be a domain whose closure is contained in an open hemisphere. Let H_Γ be the mean curvature of $\Gamma = \partial\Omega$ as a submanifold of Ω, computed with respect to the unit normal pointing to the interior of Ω. Let φ be a smooth function on $\partial\Omega$ such that*

$$H_\Gamma(q) \geq -2He^{\varphi(q)} > 0 \tag{8.27}$$

on $\partial\Omega$. Then there exists a unique solution of (8.25) with $u = \varphi$ on $\partial\Omega$.

In particular, if $H_\Gamma \geq 0$, there exists a minimal radial graph on Ω for arbitrary values of φ on $\partial\Omega$. When the prescribed boundary is included in $\mathbb{S}^2$, i.e., $\varphi = 0$, the mean curvature condition (8.27) can be relaxed. We now state the main result in this section [Lop03a]:

Theorem 8.5.6 *Let Ω be a domain on the unit sphere $\mathbb{S}^2$ whose closure is contained in an open hemisphere and let $H \leq 0$. If*

$$H_\Gamma \geq -H,$$

there exists a unique radial H-graph on Ω with boundary $\partial\Omega$.

Following the method of continuity, we introduce a real parameter $t \in [0, 1]$ into the boundary-value problems $P(tH, 0)$. We need to find C^1-estimates for a prospective solution of (8.25)–(8.26).

Let us begin by obtaining a priori bounds for $|u|$ on Ω. If $p = e^{u(q)}q$, $q \in \Omega$, denotes the position vector of a radial graph M, then $|p| = e^u$. Thus C^0-estimates

of u correspond with bounds of $|p|$ for each $p \in M$. We study the general case of boundary data φ in the Dirichlet problem and set

$$m = \min_{p\in\partial M} |p|^2, \qquad R = \max_{p\in\partial M} |p|^2.$$

Theorem 8.5.7 *Let Ω be a domain in $\mathbb{S}^2$ and let M be a radial H-graph on Ω.*

1. *If $H < 0$, then*

$$\frac{-1+\sqrt{1+mH^2}}{-H} \le |p| \le \sqrt{R}, \quad p \in M. \tag{8.28}$$

2. *If $H > 0$, then*

$$|p| \le \frac{1+\sqrt{1+RH^2}}{H}, \quad p \in M. \tag{8.29}$$

3. *If $H = 0$, then*

$$|p| \le \sqrt{R}, \quad p \in M. \tag{8.30}$$

Proof In view of (2.23) and (2.25), we have

$$\Delta\left(\frac{H}{2}|p|^2 + \langle N(p), p\rangle\right) = \big(2H^2 - |\sigma|^2(p)\big)\langle N(p), p\rangle \ge 0.$$

The maximum principle gives

$$\frac{H}{2}|p|^2 + \langle N(p), p\rangle \le \max_{p\in\partial M}\left(\frac{H}{2}|p|^2 + \langle N(p), p\rangle\right). \tag{8.31}$$

Suppose now $H < 0$. Then

$$\frac{H}{2}|p|^2 + \langle N(p), p\rangle \le \frac{H}{2}m,$$

that is, $-H|p|^2 + 2|p| + Hm \ge 0$. This quadratic inequality on $|p|$ has one negative and one positive root. Then one derives the left-hand inequality in (8.28). If $H > 0$, the argument is similar obtaining (8.29). If $H \le 0$, the maximum principle for Eq. (2.23) yields directly the right-hand inequalities in (8.28) and (8.30). □

If $\partial M \subset \mathbb{S}^2$ and $H < 0$, the inequalities (8.28) read as

$$\frac{-1+\sqrt{1+H^2}}{-H} \le |p| \le 1 \tag{8.32}$$

for all $p \in M$.

Observe that the estimates (8.28), (8.29) and (8.30) are sharp as shown in the following example:

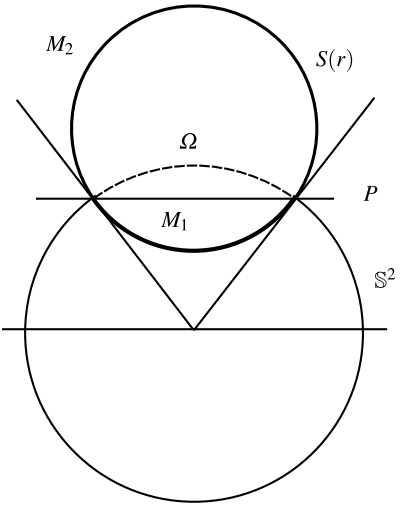

Fig. 8.8 The radial graphs M_1 and M_2 on Ω attain the extremal values of the modulus function $|p|$ given in Theorem 8.5.7

Example 8.5.8 Let $r \in (0, 1)$ and $\Omega = \{(x, y, z) \in \mathbb{S}^2 : z > \sqrt{1 - r^2}\}$. Let

$$S(r) = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + \left(z - \frac{1}{\sqrt{1 - r^2}}\right)^2 = \frac{r^2}{1 - r^2}\right\}.$$

The surface $S(r)$ is a sphere centered at the point $(0, 0, 1/\sqrt{1 - r^2})$ of radius $r/\sqrt{1 - r^2}$. Then $\partial\Omega$ determines two spherical caps on $S(r)$, namely M_1 and M_2, which are radial graphs on Ω: see Fig. 8.8. According to the choice of N, H is negative on M_1 and positive on M_2. It is easy to see that M_1 and M_2 attain the bounds (8.28) and (8.29) of Theorem 8.5.7, respectively. Notice that both graphs are tangent to the cone from the origin of $\mathbb{R}^3$ and based on $\partial\Omega$. Let us observe that M_i are orthogonal to $\mathbb{S}^2$ along its boundary. In this sense, there is some analogy with vertical graphs of $\mathbb{R}^3$, see Theorem 2.3.5. Finally, the planar domain of the plane P with equation $z = \sqrt{1 - r^2}$ and defined by $|p| \leq 1$ is a radial minimal graph on Ω and $|p| \leq 1$.

If $H > 0$, it is possible to have a priori lower estimates for $|u|$, provided the closure of Ω is included in an open hemisphere. This improves the estimate (8.29). Indeed, without loss of generality, assume that $\overline{\Omega}$ is included in the upper hemisphere of $\mathbb{S}^2$. Write $\mu = \min_{q \in \partial\Omega} z(q) > 0$. Compare M with horizontal planes arriving from the half-space $z < 0$. As the orientation on M points in the opposite direction of the vector position, the tangency principle implies that the first contact point between the planes and M occurs at boundary points and we arrive at the height $z = \mu$. In particular, $\mu \leq z(p) \leq |p|$, for all $p \in M$, and thus, $\log \mu \leq u$ on $\overline{\Omega}$.

Having obtained estimates of $|u|$ in Theorem 8.5.7, we proceed to estimate $|\nabla u|$. In order to do this, it will be convenient to express ∇u in terms of the support

function of M. The expression of N on M given in (8.24) yields

$$\langle N(p), p\rangle = -\frac{e^{u(q)}}{\sqrt{1+|\nabla u|^2(q)}}. \tag{8.33}$$

Thus the bounds of $|\nabla u|$ are obtained provided that we have estimates for the support function $\langle N, x\rangle$ together with the C^0-estimates derived for u in Theorem 8.5.7. Here we follow the same ideas as the second proof of Theorem 8.3.1.

As usual, we first assume $H_\Gamma > -H$ and at the end of the section, we return to the general case $H_\Gamma \geq -H$. In the next result, we suppose $u = 0$ on $\partial\Omega$.

Theorem 8.5.9 *Let $\Omega \subset \mathbb{S}^2$ be a domain and let $H \leq 0$. Assume that $H_\Gamma > -H$. Then there exists a positive constant $C = C(\Omega, H)$ such that if u is a solution of (8.25)–(8.26), we have*

$$\sup_{\overline{\Omega}} |\nabla u| \leq C.$$

Proof Assume that $M = \text{graph}(u)$. In view of (8.28), we have $|p| \leq 1$ on M and the maximum principle gives $\langle \nu(p), p\rangle \leq 0$ along $\partial\Omega$, where ν stands for the inner unit conormal vector of M along $\partial\Omega$. Inequality (8.31) guarantees that the function $H|p|^2/2 + \langle N(p), p\rangle$ assumes its maximum at a point $q \in \partial\Omega$:

$$\frac{H}{2}|p|^2 + \langle N(p), p\rangle \leq \frac{H}{2} + \langle N(q), q\rangle \tag{8.34}$$

for all $p \in M$. Moreover, the maximum principle gives ($\nu_q = \nu(q)$)

$$H\langle \nu_q, q\rangle + \langle dN_q \nu_q, q\rangle \leq 0.$$

Consequently,

$$\big(H - \sigma(\nu_q, \nu_q)\big)\langle \nu_q, q\rangle \leq 0.$$

This inequality and the fact that $\langle \nu_q, q\rangle \leq 0$ yields $\sigma(\nu_q, \nu_q) \leq H$. Let α be an arc length parametrization of ∂M. As $2H = \sigma(\alpha', \alpha') + \sigma(\nu, \nu)$, it follows that $H \leq \sigma(\alpha', \alpha')$. Decompose the second fundamental form σ of the immersion $\partial\Omega \hookrightarrow \mathbb{S}^2$ as the sum of the second fundamental form σ^Ω of $\partial\Omega \hookrightarrow \Omega$ and the second fundamental form of $\Omega \hookrightarrow \mathbb{S}^2$. Thus

$$\sigma\big(\alpha'(q), \alpha'(q)\big) = \sigma^\Omega\big(\alpha'(q), \alpha'(q)\big)\langle \nu_q, q\rangle - \langle N(q), q\rangle.$$

As

$$\langle \nu_q, q\rangle^2 + \langle N(q), q\rangle^2 = |q|^2 = 1$$

and $\langle \nu_q, q\rangle < 0$, we have $\langle \nu_q, q\rangle = -\sqrt{1 - \langle N(q), q\rangle^2}$. Then

$$\langle N(q), q\rangle + H \leq -H_\Gamma(q)\sqrt{1 - \langle N(q), q\rangle^2}.$$

Since $\langle N(q), q\rangle \leq 0$, we find that

$$\big\langle N(q), q\big\rangle \leq \frac{-H - H_\Gamma(q)\sqrt{H_\Gamma^2(q) + 1 - H^2}}{1 + H_\Gamma^2(q)}.$$

Let introduce the values

$$\kappa_0 = \min_{q\in\partial\Omega} H_\Gamma(q), \qquad \kappa_1 = \max_{q\in\partial\Omega} H_\Gamma(q). \tag{8.35}$$

Taking into account this inequality and (8.34), it follows that for each $p \in M$, we have

$$\begin{aligned}\big\langle N(p), p\big\rangle &\leq \frac{H}{2}\big(1 - |p|^2\big) - \frac{H + H_\Gamma(q)\sqrt{H_\Gamma(q)^2 + 1 - H^2}}{1 + H_\Gamma(q)^2}\\ &\leq -\frac{H + H_\Gamma(q)\sqrt{H_\Gamma(q)^2 + 1 - H^2}}{1 + H_\Gamma(q)^2}\\ &\leq -\frac{H + \kappa_0\sqrt{\kappa_0^2 + 1 - H^2}}{1 + \kappa_1^2} =: b_1. \end{aligned}\tag{8.36}$$

Let us remark that the constant b_1 is negative and it depends only on Ω and H. According to (8.28) and (8.33), the number b_1 determines a constant $C(\Omega, H)$ such that $|\nabla u| \leq C$. □

Observe that if $H \leq 0$, Eq. (2.25) leads to $\Delta\langle N(p), p\rangle \geq 0$. By the maximum principle, the function $\langle N(p), p\rangle$ assumes its maximum at a point of ∂M. In this case (with no assumption on H_Γ), $\sup_{\overline{\Omega}} |\nabla u| = \max_{\partial\Omega} |\nabla u|$.

From Theorems 8.5.7 and 8.5.9, we have obtained a priori C^1-estimates for solutions of (8.25)–(8.26). Once such estimates have been established, we shall prove the existence of a solution to the Dirichlet problem by means of the continuity method as usual. Define

$$\mathscr{S} = \big\{t \in [0, 1] : \text{there exists a solution } u_t \text{ of } P(tH, 0)\big\}.$$

In this context, we show that $\mathscr{S}$ is a non-empty, open and closed subset of $[0, 1]$.

First, we prove that $0 \in \mathscr{S}$. As we observed earlier, this is a consequence of Theorem 8.5.5. This means that we can start the existence procedure. In a second step, we prove that $\mathscr{S}$ is open in $[0, 1]$. Recall that the linearization of the mean curvature is the Jacobi operator $L = \Delta + |\sigma|^2$. Using (2.25), the kernel of L is trivial because

$$L\big(\langle N, x\rangle\big) = -2H \geq 0$$

with $\langle N, x\rangle < 0$ and so, L is an elliptic operator of index zero (see Remark 8.1.5).

It remains to see that $\mathscr{S}$ is closed in $[0,1]$. This is equivalent to finding global C^1-estimates of each solution u_t independent of t. As a consequence of Theorem 8.5.7 and Proposition 8.5.3, for each solution u_t, $t \in [0,1]$, we have

$$\frac{-1+\sqrt{1+H^2}}{-H} \leq \exp(u_t) \leq 1.$$

Thus

$$\log \frac{-1+\sqrt{1+H^2}}{|H|} \leq u_t \leq 0,$$

obtaining the desired a priori C^0-bounds for u_t independent of t.

Returning to estimates for $|\nabla u|$, we apply Theorem 8.5.9. We need to estimate b_1 in (8.36) from above by a bound independent of t. As $H < 0$, we have

$$b_1 = -\frac{tH + \kappa_0\sqrt{\kappa_0^2 + 1 - t^2H^2}}{1+\kappa_1^2} \leq -\frac{H + \kappa_0\sqrt{\kappa_0^2 + 1 - H^2}}{1+\kappa_1^2} < 0,$$

giving the corresponding estimates of $|\nabla u|$. Since we obtain $\sup_{\overline{\Omega}} |\nabla u| < \infty$, the function u is smooth in $\overline{\Omega}$.

Finally, we consider the case $H_\Gamma \geq -H$. Let $H_n > 0$ satisfy $H_n \searrow H$. Then $H_\Gamma \geq -H > -H_n$. By the above case, let $u_n \in C^\infty(\overline{\Omega})$ be a solution of the Dirichlet problem (8.25)–(8.26) for H_n. By Proposition 8.5.3 and Theorem 8.5.7, u_n is an increasing function with $u_n \leq 0$. By compactness arguments, there exists a subsequence, renamed u_n, and $u \in C^2(\Omega)$ such that $u_n \to u$ uniformly on compact subsets of Ω. Then u solves (8.25) for H in Ω.

Now we prove that u extends continuously to $\partial\Omega$ with $u = 0$ on $\partial\Omega$. As barriers we use the spherical caps that appeared in Example 8.5.8, which play the same role as hemispheres in Theorem 8.3.1. More precisely, let κ_0 be the number given in (8.35). Take the surface $M_1 \subset S(r)$ given in Example 8.5.8 for $r = 1/\sqrt{1+\kappa_0^2}$ and denote by Ω_1 the round disk bounded by ∂M_1 in the upper half-sphere of $\mathbb{S}^2$ (see Fig. 8.8). According to the orientation, the mean curvature of M_1 is $H_1 = -\sqrt{1-r^2}/r = -\kappa_0$ and $H_1 \leq H$. On the other hand, as $\kappa_0 \geq -H$, $\Omega \subset \Omega_1$.

We claim that $int(M_1) \cap int(G_n) = \emptyset$ where G_n if the radial graph of u_n. For this, we apply homotheties tM_1 for $t \in (0,1)$ and $t \searrow 0$ until $tM_1 \cap G_n = \emptyset$. This holds because $\Omega \subset \Omega_1$ and G_n and M_1 are radial graphs on Ω and Ω_1 respectively. Next, we go back by letting $t \nearrow 1$. If there is a first touching point between $t_0 M_1$ and G_n, with $t_0 < 1$, this contact must occur at an interior point. Thus, both surfaces are tangent. The comparison principle implies that $H_1/t_0 \geq H_n$, which is a contradiction since $t_0 < 1$ and $H_1 \leq H < H_n$. Therefore we return to the initial position of M_1 proving the claim.

This proves that G_n lies in the domain $W \subset \mathbb{R}^3$ bounded by $M_1 \cup \Omega_1$. Similar reasoning proves that this also happens for the surface M, the graph of u. We are now in position to prove the continuity of u on $\partial\Omega$. Let $p \in \partial\Omega$. Since the geodesic

curvature of $\partial\Omega_1$ is (constant) κ_0 and the geodesic curvature of $\partial\Omega$ at p is $H_\Gamma(p)$ with $\kappa_0 \leq H_\Gamma(p)$, after a rigid motion it is possible to position M_1 so that $\Omega \subset \Omega_1$ and $p \in \partial\Omega \cap \partial\Omega_1$. Let v_p be the function defined on Ω_1 whose radial graph is M_1. By the claim, $v_p < u_n$ in Ω and, in addition, we know that $u_n \leq 0$. If $p_m \to p$ with $p_m \in \Omega$, then $v_p(p_m) < u_n(p_m) \leq 0$. By letting $n \to \infty$, $v_p(p_m) \leq u(p_m) \leq 0$. Now let $m \to \infty$. By the continuity of v_p on $\overline{\Omega} \subset \overline{\Omega_1}$ and since $v_p(p) = 0$ we finally prove that $\lim_{m\to\infty} u(p_m) = 0$.

Chapter 9
The Dirichlet Problem in Unbounded Domains

This chapter deals with the Dirichlet problem for the constant mean curvature equation in the case that the domain $\Omega \subset \mathbb{R}^2$ is not bounded. There are differences between the cases $H = 0$ and $H \neq 0$. Bernstein's theorem establishesthat the only minimal graph defined on $\mathbb{R}^2$ is an affine plane. In contrast, if $H \neq 0$, there do not exist H-graphs on $\mathbb{R}^2$, in fact, Ω cannot be 'large' in the sense that Ω cannot contain in its interior a closed round disk of radius $1/|H|$. If $H = 0$, any unbounded domain of a plane is a solution of the minimal surface equation. Furthermore, uniqueness does not hold even for simple boundary conditions. For example, if $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$, the linear function $u_c(x, y) = cx$, $c \in \mathbb{R}$, satisfies the minimal surface equation on Ω for every constant c, with $u_c = 0$ on $\partial\Omega$. For unbounded domains, a pioneering article on minimal graphs is that of P. Collin and R. Krust [CK91]. See also [Ber51, ER89, EsRi11, FM11, Hwa88, Hwa94, KT98, Kuw93, LR88, Maz04a, Maz04b, Spr02].

In this chapter we shall treat the question of existence when $H \neq 0$. We shall characterize the solvability of the Dirichlet problem when Ω is convex. In this setting, we use cylinders as barriers. If Ω is not convex, we use nodoids to obtain results of existence when Ω satisfies a uniform R-circle exterior condition in the same spirit as Sect. 8.4. In addition, we shall obtain height estimates for cmc graphs with the same flavor as those obtained for bounded domains. The method of existence that we apply is a modified version of the classical Perron method of super and subsolutions. The subsolution will be a solution of the minimal surface equation, while a supersolution is replaced by a family of local upper barriers, which will be pieces of cylinders or nodoids depending on the domain. See [Jin06, Jin08] for a similar setting with variable mean curvature in unbounded domains.

The main results in this chapter appear in [Lop01b, Lop02b, Mee88, RR10].

9.1 Height Estimates

Examples of H-graphs on unbounded domains are some pieces of half cylinders. The half cylinder of radius r given by $C_r = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = r^2, z \geq 0\}$

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_9,
© Springer-Verlag Berlin Heidelberg 2013

is the graph of the function $z(x, y) = \sqrt{r^2 - y^2}$ defined on the strip $B_r = \{(x, y) \in \mathbb{R}^2 : |y| < r\}$. In this example, $z \in C^2(B_r)$ and z extend continuously up to ∂B_r. The mean curvature is $H = 1/(2r)$ with respect to the downward orientation.

R. Finn derived a necessary condition for the existence of a solution on a strip for arbitrary boundary conditions [Fin65].

Proposition 9.1.1 *Let $H \neq 0$. Consider a strip $\Omega \subset \mathbb{R}^2$ of width $w > 0$. If there is a solution of (8.1) on Ω, then $w \leq 1/|H|$.*

Proof Assume $\Omega = \{(x, y) \in \mathbb{R}^2 : -w/2 < y < w/2\}$. For each $n \in \mathbb{N}$, define the truncated domains $\Omega_n = \{(x, y) \in \Omega : -n < x < n\}$. On Ω_n, the divergence theorem applied to (8.1) gives

$$4nw|H| = 2|H| \operatorname{area}(\Omega_n) \leq L(\partial\Omega_n) = 4n + 2w.$$

Letting $n \to \infty$, we conclude $|H| \leq 1/w$. □

This result extends to unbounded convex domains.

Theorem 9.1.2 *Let $H \neq 0$ and let $\Omega \subset \mathbb{R}^2$ be an unbounded convex domain. If (8.1) has a solution, then Ω is included in a strip of width $1/|H|$.*

Proof Let u be a solution of (8.1) defined on Ω and let G denote the graph of u. Consider the orientation given in (8.3). Since Ω is not bounded, we cannot directly apply Corollary 3.1.7 and so we do not know the sign of H. In view of Theorem 3.3.2, for each $p \in \Omega$, the closed disk $D(p, 1/|H|)$ is not included in Ω. Using this property and the convexity of Ω, it follows that Ω is included in a strip B' of width $2/|H|$ and by means of a rigid motion, assume that

$$B' = \left\{(x, y) \in \mathbb{R}^2 : -\frac{1}{|H|} < y < \frac{1}{|H|}\right\}. \tag{9.1}$$

Since Ω is a convex domain, there are two possibilities. First, if the x-coordinate of all points of Ω is not bounded from above and from below, then Ω is a strip included in B'. Then Proposition 9.1.1 shows that the width of Ω is less than $1/|H|$.

In the other case, the x-coordinate is bounded from below or from above, and the two branches of $\partial\Omega$ are asymptotic to a strip $B'_h \subset B'$ of width h, each being a branch asymptotic to each straight line of $\partial B'_h$. After a translation, suppose that the infimum of the x-coordinate of Ω is 0. Then $\Omega \subset \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Furthermore, $\partial\Omega \to \partial B'_h$ as $x \to +\infty$. By the convexity of Ω and because Ω is not bounded, for each $n > 0$, consider the two points $\{y_1(n), y_2(n)\} = \Omega \cap \{(n, y) : y \in \mathbb{R}\}$, with $y_1(n) < y_2(n)$. For each $m > n > 0$, let

$$\Omega_{n,m} = \left\{(x, y) \in \mathbb{R}^2 : n < x < m, y_1(n) < y < y_2(n)\right\}$$

be the rectangle whose sides have lengths $m - n$ and $y_2(n) - y_1(n)$. See Fig. 9.1. The convexity of Ω implies that $\Omega_{n,m} \subset \Omega$.

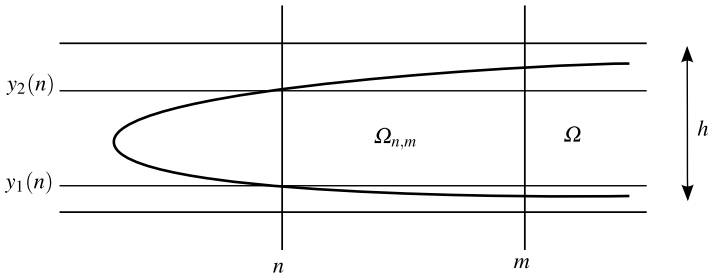

Fig. 9.1 The domains $\Omega_{n,m}$ in the proof of Theorem 9.1.2

An integration of (8.1) on $\Omega_{n,m}$ and the divergence theorem yields

$$\begin{aligned} 2|H|(m-n)\big(y_2(n)-y_1(n)\big) &= 2|H|\,\mathrm{area}(\Omega_{n,m}) \\ &= \left|\int_{\partial\Omega_{n,m}} \frac{\langle \nabla u, \mathbf{n}\rangle}{\sqrt{1+|\nabla u|^2}}\right| < L(\partial\Omega_{n,m}) \\ &= 2\big(m-n+y_2(n)-y_1(n)\big) \le 2\left(m-n+\frac{2}{|H|}\right), \end{aligned}$$

where $\mathbf{n}$ denotes the outward unit normal vector along $\partial\Omega_{n,m}$. Letting $m\to\infty$, we see $y_2(n)-y_1(n)\le 1/|H|$. Since the two branches of $\partial\Omega$ are asymptotic to the strip $\partial B'_h$, in the limit case $n\to\infty$, we conclude $h\le 1/|H|$ and the theorem is proved. □

Constant mean curvature graphs defined on unbounded domains with $\varphi = 0$ in the boundary condition (8.2) have similar height estimates as in Theorem 2.3.5. We point out that, although many of the results on existence of the Dirichlet problem in this chapter are formulated for a convex domain Ω, the next theorem holds for arbitrary domains Ω.

Theorem 9.1.3 *Let $H>0$. Consider an unbounded domain $\Omega\subset\mathbb{R}^2$ and let $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$ with $u=0$ on $\partial\Omega$. If $M=\mathrm{graph}(u)$ is an H-graph, then $0<|u|\le 1/H$. In particular, either $u>0$ on Ω or $u<0$ on Ω.*

Proof Denote by N the orientation corresponding to $H>0$ and let P_t be the horizontal planes with equation $z=t$. First we prove that M lies on one side of P_0. Assume that u takes negative and positive values in Ω. Let $q\in\Omega$ be a point such that $u(q)>0$ and consider the vertical half-line $L=\{q\}\times[u(q),\infty)$. Consider the spheres of radius $1/H$ centered at L and approaching from above. In the first contact point with M, the intersection must occur at a point p whose third coordinate is greater than or equal to $u(q)$. By the tangency principle, $N(p)$ does not coincide with the unit normal vector of the sphere at p. As M is a graph, the point p must necessarily belong to the lower hemisphere. Since the unit normal vector of the sphere (for $H>0$) points inside, then at p it points upwards. It follows that

$N(p)$ points downwards. As M is a graph, $\langle N, e_3\rangle < 0$ on M. Now, we proceed with points where $u < 0$ and comparing with spheres approaching from below. A similar argument and the tangency principle prove that M and the first sphere of contact agree on a common open set which implies, appealing to Theorem 2.2.2, that M is umbilical, a contradiction. In fact, if u takes positive values, then $u > 0$ in Ω and $int(M) \cap P_0 = \emptyset$: on the contrary, and if $q \in int(M) \cap P_0$, the comparison principle between M and P_0 at q yields a contradiction because P_0 lies above M around q. This proves that $u \neq 0$ on $int(M)$, which is the first statement of theorem.

Without loss of generality, assume that $u > 0$ in Ω. The previous reasoning proves that if $H > 0$, the orientation N points downwards. The proof of theorem comprises two steps.

1. *Claim*. The height of M is bounded.
   This result is particular case of a more general result that asserts that given a strong stable H-surface, the intrinsic distance from a point of M to ∂M is less than or equal to π/H ([RR10]; also [FS80, LR89]). In this case, an H-graph is strong stable by Theorem 2.3.10. As $u = 0$ on $\partial\Omega$, the value $|u(q)|$ is less than the distance between $(q, u(q))$ and ∂M, which is less than or equal to π/H, proving that u is bounded.
   However, we present a proof which does not use this result. Here we follow [Mee88] with some minor modifications. As $u > 0$ in Ω, let W be the domain bounded by $M \cup \Omega$ that lies below M. Consider the plane $P_{1/H}$. If $P_{1/H} \cap M = \emptyset$, the height of M is bounded by $1/H$ and the claim is proved. Thus, we suppose that $P_{1/H}$ does intersect M and that this intersection is transverse: if not, we replace $P_{1/H}$ by another horizontal plane $P_{\epsilon+1/H}$, $\epsilon > 0$ where the intersection is transverse. Let D be a connected component of $P_{1/H} \cap W$ and assume that there are two points $q_1, q_2 \in D$ at distance greater than $2/H$. Then there are two parallel lines L_1 and L_2 contained in $P_{1/H}$ and at distance $2/H$ apart, with $q_i \in L_i$, $i = 1, 2$. Let $\alpha : [0, 1] \to D$ be an embedded arc with $\alpha(0) = \alpha([0, 1]) \cap L_1$ and $\alpha(1) = \alpha([0, 1]) \cap L_2$ (the end points of α may not agree with the points q_i). Let L_1', L_2' be the vertical translations of L_i in the plane P_0. If $\alpha(s) = (x(s), y(s), 1/H)$, let $E = \{(x(s), y(s), t/H) : s, t \in [0, 1]\}$ denote the set of all vertical translations of α between the heights $z = 0$ and $z = 1/H$. Then E is included in W because M is a graph on Ω and $\alpha([0, 1]) \subset W$. Consider the convex hull Q of $L_1 \cup L_2 \cup L_1' \cup L_2'$. Since $E \subset Q$ is embedded with $\partial E \subset \partial Q$, the set $Q \setminus E$ has exactly two connected components, and denote by Q_1 and Q_2 the closures of both components. The surface E will be used as 'a door' to enter into the domain W with a hemisphere of radius $1/H$. See Fig. 9.2.
   Let $L \subset P_0$ denote the equidistant line from L_1' and L_2' and, after a horizontal translation and a rotation about the z-axis, assume that $L = \{(t, 0, 0) : t \in \mathbb{R}\}$. Suppose that Q_1 is the connected component that intersects L for values t close to $t = -\infty$. Consider the 1-parameter family of hemispheres $\{S_t : t \in \mathbb{R}\}$ centered at $(t, 0, 0)$ of radius $1/H$ with $\partial S_t \subset P_0$ and contained in $\{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$. Then $S_t \subset Q$ for all t, and for t close to $-\infty$, S_t is included in Q_1. Increase t until S_{t_0} first contacts with E at $t = t_0$. Next we continue to increase t

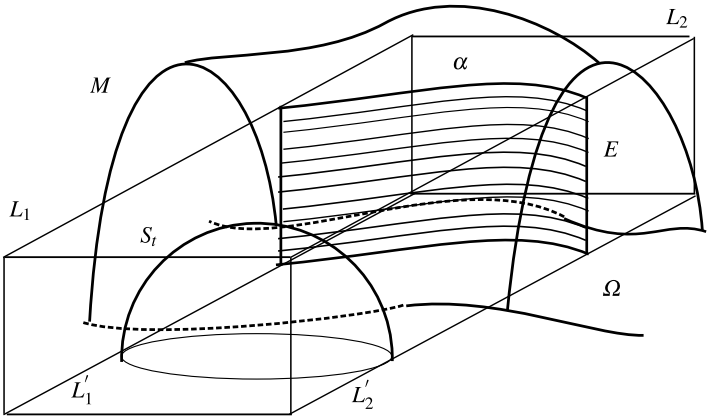

Fig. 9.2 Proof of the boundedness of an H-graph

and let $S_t^+ = S_t \cap Q_2$. Note that for some $\epsilon > 0$,

$$\partial S_t^+ \subset E \cup (\Omega \cap Q_2) \quad \text{and} \quad S_t^+ \subset \overline{W}$$

for $t \in (t_0, t_0 + \epsilon)$. We point out that the part of ∂S_t^+ in $\Omega \cap Q_2$ is a piece of a circle of radius $1/H$.

We claim that for some $t_1 > t_0$, $S_{t_1} \subset \overline{W}$. On the contrary, there exists a first time $t_2 > t_0$ such that $S_{t_2}^+ \subset \overline{W}$, $S_{t_2+\delta}^+ \not\subset \overline{W}$ for $\delta > 0$. As $\partial \overline{W} = M \cup \Omega$, it follows that $S_{t_2}^+ \setminus (\partial S_{t_2}^+ \cap E)$ intersects M at a point $p \in M$. If p is an interior point, both surfaces are tangent; if $p \in \partial M$, both surfaces are tangent again because $S_{t_2}^+$ is orthogonal to P_0 along $\partial S_{t_2}^+ \cap P_0$, which means M is vertical at the contact point. Since $S_{t_2}^+$ has mean curvature H with the inward orientation and $S_{t_2}^+ \geq M$ around the contact point, the tangency principle implies that $S_{t_2}^+ \subset M$ in an open set around p. In particular, there is an open set in M formed by umbilical points and by Theorem 2.2.2, M is included in a sphere, a contradiction. Having proved this claim, and since $S_{t_1} \subset \overline{W}$, we conclude that Ω contains the closure of a disk of radius $1/H$, which is impossible by Theorem 3.3.2.

Summarizing, we have proved that the diameter of each connected component of $P_{1/H} \cap W$ is at most $2/H$, in particular, such a connected component is bounded. Thus, and in view of Theorem 2.3.5, the part of M above $P_{1/H}$, which is an H-graph on bounded domains, has a height of at most $1/H$. This proves that the height of M is less than $2/H$ and so, is bounded. We point out that if we need to work with the intersection $P_{\epsilon+1/H} \cap M$, the same argument proves that the part over $P_{\epsilon+1/H}$ is bounded. In fact, the height of this part is less than $1/H$, but letting $\epsilon \to 0$ and by transversality, the height of M over $P_{1/H}$ is less than or equal to $1/H$.

2. *Claim.* $u \leq 1/H$.
   Consider on M the function $f = H\langle x, e_3\rangle + \langle N, e_3\rangle$. If $f \leq 0$ on M, then $\langle x, e_3\rangle \leq -\langle N, e_3\rangle/H \leq 1/H$ and the height estimate is proved. Thus, assume

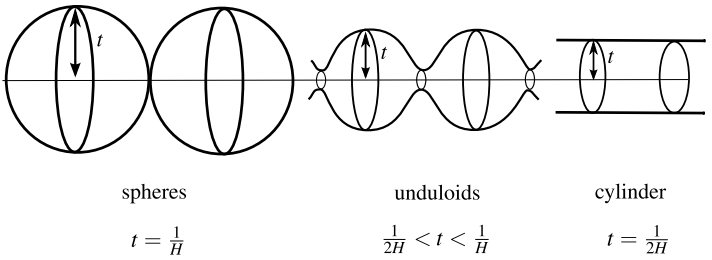

Fig. 9.3 Unduloids with the same mean curvature H and parameterized by the bulge radius t. As the parameter t ranges from $t = 1/H$ to $t = 1/(2H)$, the H-surfaces change from a stack of spheres, through to unduloids and, finally, to a cylinder

that f is positive at some point. By the preceding step, we know that $\langle x, e_3\rangle$ is bounded. Let $c = \sup_M f$ and take $p_n \in M$ such that $f(p_n) \to c$. Taking a subsequence if necessary, we assume that $u(p_n) \to z_0$ and $\langle N(p_n), e_3\rangle \to n_0$. If $z_0 \leq 0$, then $c = \lim f(p_n) = Hz_0 + n_0 \leq 0$, a contradiction. Thus, $z_0 > 0$. Then we assume that $u(p_n) > 0$ for all n.

For each n, displace horizontally M so that p_n lies in the z-axis and denote by M_n the new surface. In view of Lemma 6.4.1, we know that an H-graph with zero boundary values satisfies a priori interior curvature estimates. Therefore a subsequence of M_n converges to an H-graph on a domain of $\mathbb{R}^2$ and the boundary of this graph lies in P_0 because $\partial M_n \subset P_0$. Let $\bar{p} = \lim p_n$ and $\widetilde{M}$ be the connected component of the limit surface containing $\bar{p}$. As $z(\bar{p}) = \lim u(p_n) = z_0 > 0$, we have $\bar{p} \in int(\widetilde{M})$. Since the function f defined on $\widetilde{M}$ assumes a maximum at $\bar{p}$ and $\Delta f \geq 0$, it follows that f is constant on $\widetilde{M}$. By Lemma 2.3.4, $\widetilde{M}$ is umbilical. As $H \neq 0$ and $\widetilde{M}$ is a graph, $\widetilde{M}$ is part of a small H-spherical cap with boundary on P_0. But in this cap, $f \leq 0$, contradicting the fact that $c > 0$. □

The height estimate obtained in this theorem is sharp because there exist unduloids that are H-graphs on unbounded domains whose height is as close to $1/H$ as we wish. Unduloidsare embedded cmc surfaces of revolution and translationally periodic [Del41]. Place an H-unduloid so that its axis of revolution L lies in the plane $z = 0$. The unduloid has (infinite) bulges and necks, in fact, this unduloid belongs to a 1-parameter family of H-unduloids $\{U_t : 1/(2H) < t < 1/H\}$ with the same axis L and the parameter t indicates the bulge radius. If $t \to 1/H$, the family converges to a stack of spheres of radius $1/H$ whose centers lie in L and if $t \to 1/(2H)$, the limit surface is a cylinder with axis L and radius $1/(2H)$. See Fig. 9.3. The part U_t^+ of U_t above P_0 is the graph of a function u_t in a domain $\Omega_t \subset P_0$, $u_t = 0$ along $\partial\Omega_t$ and U_t^+ is vertical along $\partial\Omega_t$. Then the height of U_t^+ ranges from $1/H$ (spheres) to $1/(2H)$ (cylinder).

A direct consequence of the proof of Theorem 9.1.3 is:

Corollary 9.1.4 *Let $H > 0$. Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$. If φ is bounded and $M = graph(u)$ is an H-*

graph, *then*

$$\inf_{\partial\Omega}\varphi - \frac{1}{H} \le u \le \sup_{\partial\Omega}\varphi + \frac{1}{H}.$$

This corollary extends the estimates obtained in Theorem 2.3.5 for unbounded domains of $\mathbb{R}^2$.

9.2 Graphs on Convex Domains

The next theorem provides a characterization of the solvability of the Dirichlet problem when Ω is an unbounded convex domain and $u = 0$ along $\partial\Omega$.

Theorem 9.2.1 *Let $H \neq 0$ and let $\Omega \subset \mathbb{R}^2$ be an unbounded convex domain included in a strip of width $1/|H|$. Then there exists a solution u of the Dirichlet problem $P(H, 0)$. Moreover $0 < |u| \le 1/(2|H|)$ in Ω.*

Proof Observe that if u is a solution of $P(H, 0)$, the function $-u$ is a solution of $P(-H, 0)$. Thus we may prescribe the sign of H and, without loss of generality, suppose $H > 0$ with the downwards orientation. This means that $u > 0$ in Ω. Consider Ω included in the strip $B = \{(x, y) \in \mathbb{R}^2 : -1/(2H) < y < 1/(2H)\}$.

By Corollary 9.1.4 we know that there are height estimates for a solution u of $P(H, 0)$. Recall that these estimates hold for any domain of $\mathbb{R}^2$, independently of whether Ω is or is not convex. However, in the case that we are considering here, that is, where Ω is included in a strip of width $1/H$, a standard comparison argument with half cylinders of the type $z(x, y) = \sqrt{1/(4H^2) - y^2}$ and the plane $z = 0$ gives us

$$0 < u(x, y) \le z(x, y), \quad (x, y) \in \Omega. \tag{9.2}$$

See also the proof of Theorem 8.3.2. This proves the last statement of the theorem.

Now we prove the existence of a solution where Ω is either an infinite strip parallel to the x-axis, or Ω is a domain where the x-coordinate of all points of Ω is bounded above or below and asymptotic to a strip included in B.

If Ω is a strip, by means a rigid motion, assume that $\Omega = B_h$ with $h \le 1/(2H)$. Then

$$u(x, y) = \sqrt{\frac{1}{4H^2} - y^2} - \sqrt{\frac{1}{4H^2} - h^2}$$

is a solution of (8.1) with $u = 0$ on $\partial\Omega$ and u describes a piece of a cylinder of radius $1/(2H)$.

Assume now that Ω is not a strip. In order to find a solution u of the corresponding Dirichlet problem, we shall employ the classical Perron method of subfunctions for (8.1) (see [CH89, Vol. 2]; also [GT01, Ser70]). This will be possible provided

that for sufficiently small disks there exist solutions of (8.1) with arbitrary continuous boundary values.

Let $v \in C^0(\overline{\Omega})$ and let $D \subset \Omega$ be a closed disk. Denote by $\overline{v}$ the unique solution of the Dirichlet problem

$$Q_{-H}(\overline{v}) = 0 \text{ in } D, \quad \overline{v} = v \text{ on } \partial D.$$

Here we use Q_{-H} instead of Q_H because we are assuming the downward orientation in (8.1). The existence of the function $\overline{v}$ is guaranteed by Theorem 8.3.7 because as $D \subset \Omega$, the radius of the disk D is less than $1/(2H)$. It follows that the curvature of ∂D is greater than $2H$. Define the continuous function $M_D(v)$ in Ω by

$$M_D(v)(p) = \begin{cases} \overline{v}(p) & p \in D \\ v(p) & p \in \Omega \setminus D. \end{cases}$$

A function $v \in C^0(\overline{\Omega})$ is said to be a *subsolution* in Ω if $v \leq M_D(v)$ for every disk D in Ω. Moreover, and in such case, v is called a *subfunction*relative to 0 if $v \leq 0$ along ∂D.

Let F denote the class all subfunctions relative to 0. First, observe that $F \neq \emptyset$ since $0 \in F$: for any disk $D \subset \Omega$, $0 \leq M_D(0)$ in D by the maximum principle. The next lemma gives some properties of F that are needed in the Perron technique.

Lemma 9.2.2

1. *If* $\{v_1, \ldots, v_n\} \subset F$, *then* $\max\{v_1, \ldots, v_n\} \in F$.
2. *If* $v \in F$ *and if* D *is a disk in* Ω, *then* $M_D(v) \in F$.

Proof

1. Let D be a disk in Ω and let $v = \max\{v_1, \ldots, v_n\}$. On ∂D, we have $v_i \leq v$ and the maximum principle gives $M_D(v_i) \leq M_D(v)$ for each i. Then $\max\{M_D(v_1), \ldots, M_D(v_n)\} \leq M_D(v)$. Furthermore, as the functions v_i are subsolutions, $v_i \leq M_D(v_i)$ on D and thus $v \leq \max\{M_D(v_1), \ldots, M_D(v_n)\}$. As a consequence, we obtain the desired inequality $v \leq M_D(v)$. This proves that v is a subsolution. On the other hand, $v_i \leq 0$ on $\partial\Omega$. Thus $v \leq 0$ on $\partial\Omega$ and we deduce that v is a subfunction in Ω relative to 0.
2. First, we show that

$$M_D(v) \leq M_{D'}\big(M_D(v)\big)$$

for any disk $D' \subset \Omega$. This clearly holds if D' lies either completely inside or completely outside of D. Thus we need only consider the case where D' is partly inside and partly outside of D. As $v \in F$, it follows that $v \leq M_D(v)$ and the maximum principle yields $\overline{v} \leq \overline{M_D(v)}$ on D'. Since $v \leq M_{D'}(v)$, the maximum principle again implies

$$v \leq M_{D'}(v) \leq M_{D'}\big(M_D(v)\big). \tag{9.3}$$

Then $M_D(v) = v \leq M_{D'}(M_D(v))$ in $D' \setminus D$. We study the situation in $D' \cap D$.

a. If $p \in \partial(D' \cap D) \cap \partial D$, then $M_D(v)(p) = v(p)$ and by (9.3), $M_D(v)(p) \leq M_{D'}(M_D(v))(p)$.
b. If $p \in \partial(D' \cap D) \cap \partial D'$, then $M_{D'}(M_D(v))(p) = M_D(v)(p)$.

As a consequence, it follows that

$$M_D(v) \leq M_{D'}\big(M_D(v)\big)$$

on $\partial(D' \cap D)$ and the maximum principle yields the desired inequality $M_D(v) \leq M_{D'}(M_D(v))$ in $D' \cap D$. Finally, since $\partial\Omega \subset \overline{\Omega} \setminus D$, $M_D(v) = v \leq 0$ on $\partial\Omega$. Hence we conclude that $M_D(v)$ is a subfunction in Ω relative to 0. □

Let z be the function defined on B given by $z(x, y) = \sqrt{1/(4H^2) - y^2}$ and $Z = z_{|\Omega}$ its restriction to Ω. Consider

$$F^* = \{v \in F : 0 \leq v \leq Z \text{ in } \Omega\}.$$

Note that the functions in F^* are uniformly bounded. Also, the functions in F^* satisfy properties (1) and (2) of Lemma 9.2.2. For property (1), $0 \leq v_i \leq Z$ and the same is true of the function $\max\{v_1, \ldots, v_n\}$. For the second property, let $v \in F^*$. As $0 \leq v \leq Z$, the maximum principle implies $M_D(0) \leq M_D(v) \leq M_D(Z)$ on D. But $M_D(Z) = Z$ and $M_D(0) \leq 0$.

As $M_D(v) \in F^*$ if $v \in F^*$ and $v \leq M_D(v)$, define

$$u = \sup\big\{v : v \in F^*\big\} = \sup\big\{M_D(v) : v \in F^*\big\}.$$

The Perron method shows that u is a solution of $Q_{-H}(u) = 0$ in Ω (see [GT01, Theorem 6.11], also Proposition 8.1.7). We prove that u is continuous in $\overline{\Omega}$ and that $u = 0$ on $\partial\Omega$. The function u is continuous in Ω by Harnack's principle: if D is a disk in Ω, the functions $M_D(v)$ are uniformly bounded and this implies uniform boundedness of their first derivatives. Thus the family $\{M_D(v) : v \in F^*\}$ is uniformly continuous in D and the same is true of the function u.

It remains to check that the solution u actually takes the value 0 on $\partial\Omega$. Here we follow similar ideas as in the proof of Theorem 8.3.2. Let $p \in \partial\Omega$. Consider an infinite quarter-cylinder C_p of radius $1/(2H)$ whose axis is parallel to the tangent line L to $\partial\Omega$ at the point p. Now position C_p in $\{(x, y, z) : z \geq 0\}$ in such a way that its axis and one component of C_p lie in the (x, y)-plane. In addition, assume that the concave side of C_p is looking towards p and that its axis lies in the connected component of $\mathbb{R}^2 \setminus L$ that does not contain Ω. By the convexity of $\partial\Omega$, we get $C_p \cap G = \emptyset$, where G is the graph of the function u. Move C_p towards Ω by horizontal translations keeping the axis of C_p parallel to L until contact with G. By the tangency principle, the convexity of Ω and the height estimate (9.2), the first contact point between the surfaces C_p and G occurs at p. Represent by v_p the function whose graph is C_p and defined on a strip B_p of width $1/(2H)$ where $p \in \partial B_p$. Consequently, the functions v_p and $\psi = 0$ are a modulus of continuity in a neighborhood of p, where $\psi < u < v_p$ on $B_p \cap \Omega$. Since $v_p(p) = \psi(p) = 0$, we obtain $u(p) = 0$. This completes the proof of Theorem 9.2.1. □

For an unbounded convex domain, the same argument proving the estimate of an H-graph gives the following consequence:

Corollary 9.2.3 *Let $H > 0$. Let $\Omega \subset \mathbb{R}^2$ be an unbounded convex domain and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$. If φ is bounded and $M = graph(u)$ is an H-graph, then*

$$\inf_{\partial\Omega} \varphi - \frac{1}{2H} \leq u \leq \sup_{\partial\Omega} \varphi + \frac{1}{2H}.$$

The solution obtained in Theorem 9.2.1 is unique [CK91]. Here the uniqueness is not a direct consequence of a 'move up–move down' argument, such as that used for the bounded case.

In view of Theorems 9.1.2 and 9.2.1, we conclude:

Corollary 9.2.4 *Let $H > 0$ and let $\Omega \subset \mathbb{R}^2$ be an unbounded convex domain. Then there exists an H-graph in Ω with boundary $\partial\Omega$ if and only if Ω is included in a strip of width $1/H$.*

When Ω is a bounded domain included in a strip, we have obtained in Theorem 8.3.2 an existence result for the Dirichlet problem $P(H, 0)$ when Ω is convex and the width of the strip is less than $1/|H|$. Therefore, Theorem 9.2.1 can be seen as an extension of this result for unbounded convex domains.

9.3 The Exterior Circle Condition

In the aforementioned paper [Fin65, p. 159], for $H > 0$, R. Finn conjectured that the half cylinder of radius $1/(2H)$ is the only H-graph in a strip of width $1/H$. However, a counterexample was provided by P. Collin in [Col90] who succeeded in proving the existence of an H-graph on a strip of width $1/H$ whose boundary consists of two equal convex functions on $\partial\Omega$ (A. N. Wang proved a similar existence result for the particular case that the boundary data is a parabola [Wan90]). Both graphs will be obtained as a consequence of Theorem 9.3.1 below.

In this section we assume that the boundary satisfies a uniform exterior circle condition. In order to obtain the a priori C^1 estimates, we use pieces of nodoids as barriers as in Theorem 8.4.3. Consider a strip $\Omega \subset \mathbb{R}^2$ of width $w > 0$ given by

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < w\}.$$

For each smooth function $f : \mathbb{R} \to \mathbb{R}$, we associate the function φ_f defined on $\partial\Omega$ by

$$\varphi_f(x, 0) = \varphi_f(x, w) = f(x).$$

A solution u of (8.1) with φ_f as boundary values is a graph on Ω spanning the curve with two connected components, namely,

$$\Gamma(f) = \big\{\big(x, 0, f(x)\big) : x \in \mathbb{R}\big\} \cup \big\{\big(x, w, f(x)\big) : x \in \mathbb{R}\big\}.$$

Then $\Gamma(f)$ is formed by two copies of the graph of f, each copy positioned over one of the straight lines that defines $\partial\Omega$. Let f be a smooth function defined on $\mathbb{R}$. We say that f *satisfies a uniform exterior R-circle condition*if such property holds for one of the two domains of $\mathbb{R}^2 \setminus \text{graph}\,(f)$.

Now we establish two theorems concerning the solvability of the Dirichlet problem (8.1)–(8.2). With the notation of Proposition 8.4.1, the first result addresses the existence of solutions of the Dirichlet problem when Ω is a strip.

Theorem 9.3.1 *Let $H > 0$ and let f be a smooth function on $\mathbb{R}$ that satisfies a uniform exterior $\rho_R(H)$-circle condition, $0 < R(H) < \infty$. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < w\}$ be a strip of width $w \le 2h_R(H)$. Then there exists a solution of $P(H, \varphi)$ with $\varphi = \varphi_f$. Moreover u satisfies*

$$u(x, y) < f(x) + \frac{1}{2H} \quad \text{or} \quad f(x) - \frac{1}{2H} < u(x, y), \quad (x, y) \in \Omega. \tag{9.4}$$

Proof First we need an existence result concerning a minimal graph on the strip. The following proposition is a consequence of a theorem of J.C.C. Nitsche [Nit65] and H. Jenkins and J. Serrin [JS63] for bounded domains with piecewise smooth convex boundary (see also [ER89]).

Proposition 9.3.2 *Let $\Omega \subset \mathbb{R}^2$ be a strip and let f be a continuous function on $\partial\Omega$. Then there exists a solution of the Dirichlet problem $P(0, \varphi_f)$.*

Proof For each $n \in \mathbb{Z}$ and $n > w$, consider the intervals $I_n = \{n\} \times (0, w)$ and $J_y = (-n, n) \times \{y\}$, $y \in \{0, w\}$. Construct an exhaustion of subdomains $\Omega_n \subset \Omega_{n+1}$ of Ω, where Ω_n is the rectangular domain bounded by the segments $I_{-n} \cup I_n \cup J_0 \cup J_w$. Since the line segments of its boundary satisfy

$$\text{length}(I_{-n}) + \text{length}(I_n) < \text{length}(J_0) + \text{length}(J_w),$$

the mentioned Nitsche-Jenkins-Serrin theorem proves the existence of a sequence v_n (resp. v'_n) of solutions of the minimal surface equation on Ω_n assuming the value f along $J_0 \cup J_w$ and $+\infty$ (resp. $-\infty$) along $I_{-n} \cup I_n$. Then v_n (resp. v'_n) is a decreasing (resp. increasing) sequence. If $D \subset \Omega_m$ is a compact set, for $n \ge m$, we have on D the inequalities:

$$\inf_D v'_m \le v'_m \le v'_n \le v_n \le v_m \le \sup_D v_m.$$

By compactness, a diagonal argument shows that v_n converges to a minimal solution ϕ on Ω. Using the Scherk surface as a barrier, we deduce $\phi = f$ along $\partial\Omega$, as desired. □

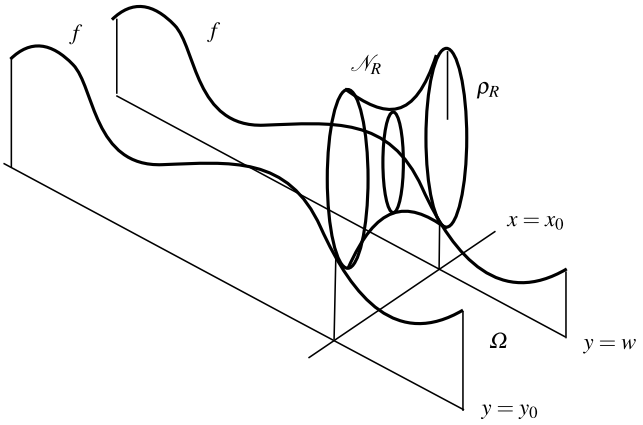

Fig. 9.4 Proof of Theorem 9.3.1. The nodoid $\mathcal{N}_R$ rolls along $\Gamma(f)$ proving that $u = f$ along $\partial\Omega$

There is no loss of generality in assuming that the domain of $\mathbb{R}^2 \setminus \text{graph}\,(f)$ that satisfies the uniform exterior ρ_R-circle condition is $\{(x, z) \in \mathbb{R}^2 : z > f(x)\}$. Let $\mathcal{N}_R$ denote the nodoid of constant mean curvature H given by Proposition 8.4.1 for the parameter $t = R$. This surface is bounded by two coaxial circles of radii ρ_R in parallel planes separated by a distance of $2h_R$.

Assuming existence, we first prove that one of the two inequalities (9.4) holds. More precisely, by the above assumption on the planar domain $z > f(x)$, we prove that $u(x, y) < f(x) + 1/(2H)$ for all $(x, y) \in \Omega$. Let u be a solution of (8.1)–(8.2) and let G be its graph. Let $(x_0, y_0) \in \Omega$. Consider the vertical planes Π_t with equations $y = t$. We know that the domains $D_0 = \{(x, 0, z) : z > f(x)\} \subset \Pi_0$ and $D_w = \{(x, w, z) : z > f(x)\} \subset \Pi_w$ satisfy the uniform exterior ρ_R-circle condition. Position $\mathcal{N}_R$ so that its axis is parallel to the y-axis and the circle C_R given by Proposition 8.4.1 lies over the line $y = w/2, z = 0$. Take the connected component obtained from $\mathcal{N}_R$ when we cut it by the planes Π_0 and Π_w, and denote this component by $\mathcal{N}_R$ again. The boundary curves of $\mathcal{N}_R$ are two circles of radii $\leq \rho_R(H)$ and located in planes parallel to Π_0, Π_w respectively. Moreover $\mathcal{N}_R$ is symmetric about the plane $y = w/2$.

Move $\mathcal{N}_R$ so that the two open disks defined by $\partial\mathcal{N}_R$ are included in $D_0 \cup D_w$ and $\partial\mathcal{N}_R$ touches the points $p_0 = (x_0, 0, f(x_0))$ and $p_w = (x_0, w, f(x_0))$. This is possible by the uniform $\rho_R(H)$-circle condition and because the radius of $\partial\mathcal{N}_R$ is less than $\rho_R(H)$. See Fig. 9.4. Now move $\mathcal{N}_R$ vertically upwards so that $\mathcal{N}_R \cap G = \emptyset$ and next, move $\mathcal{N}_R$ downwards until it first touches G. By the tangency principle and since G is a graph, the first contact with G cannot be at an interior point of G because the mean curvature vector of $\mathcal{N}_R$ points downwards (see Proposition 8.4.1(d)). Thus no tangent points exist before reaching $\Gamma(f)$ and this occurs when $\partial\mathcal{N}_R$ touches $\Gamma(f)$ at the points p_0 and p_w. Then $u(x_0, y_0) < f(x_0) + \rho_R < f(x_0) + 1/(2H)$.

Geometrically, we can visualize this as follows. By the tangency principle, descend $\mathcal{N}_R$ from $z = +\infty$ until $\mathcal{N}_R$ touches $\Gamma(f)$ and no contact points occurs

between $\mathscr{N}_R$ and G. By the uniform exterior circle condition, and because the mean curvature vector of $\mathscr{N}_R$ points outside W_R (Proposition 8.4.1(d)), we roll $\mathscr{N}_R$ along $\Gamma(f)$ so that during the displacement, the nodoid $\mathscr{N}_R$ touches G only at boundary points of G, and all points of ∂G are contacted.

Now we proceed with the proof of the existence part of Theorem 9.2.1. Let $\Omega_R \subset \Omega$ be the region onto which $\mathscr{N}_R$ projects by vertical projection. Let $p \in \partial\Omega$, $p = (x_0, \epsilon)$, where $\epsilon = 0$ or $\epsilon = w$. Position $\mathscr{N}_R$ so that one of the two points at lowest height (necessarily boundary points) projects onto p. Consider the half part of $\mathscr{N}_R$ that lies below the horizontal plane containing its axis. Denote by $\mathscr{N}_R^*$ this piece of $\mathscr{N}_R$. Then $\mathscr{N}_R^*$ is an H-graph on a domain $\Omega_p \subset \Omega$ with the unit normal vector field pointing downward. Position $\mathscr{N}_R^*$ so that the point $(p, f(x_0))$ belongs to $\partial\mathscr{N}_R^*$. Let v_p be the function on Ω_p whose graph is $\mathscr{N}_R^*$. Finally, let ϕ be the minimal solution guaranteed by Proposition 9.3.2.

Let Σ be the family of functions v_p with $p \in \partial\Omega$. On the other hand, we have $f \le v_p$ along $\partial\Omega_p \cap \partial\Omega$. Since the graph of ϕ is a minimal surface, the tangency principle gives $\phi \le M_D(\phi)$ for each disk $D \subset \Omega$, which means that ϕ is a subsolution that satisfies $\phi \le v_p$, for every $v_p \in \Sigma$.

We introduce the concept of subfunction relative to f similar to that given in Theorem 9.2.1. A function v is said to be a *subfunction relative to* f if v is a subsolution in Ω and $v \le f$ along $\partial\Omega$. We prove that between the function v_p of Σ and the subsolution ϕ, there exists a solution of (8.1). Let F_f^* be the class of all subfunctions v such that $\phi \le v \le v_p$ in Ω_p, for all $p \in \partial\Omega$.

The functions of F_f^* are uniformly bounded in each compact set of Ω. Since $\varphi \in F_f^*$, we define $u : \overline{\Omega} \to \mathbb{R}$ by

$$u(x, y) = \sup_{v \in F_f^*} v(x, y).$$

Similarly as in Lemma 9.2.2, the set F_f^* is stable by M_D, i.e., if $v \in F_f^*$, then $M_D(v) \in F_f^*$. Since $v \le M_D(v)$, we have

$$u(x, y) = \sup_{v \in F_f^*} M_D(v)(x, y).$$

Using compactness and the maximum principle, it follows that u is a solution of (8.1) just as in Theorem 9.2.1.

Finally, we employ a standard barrier argument to prove that u is a continuous function in $\overline{\Omega}$. In order to show that $u = \varphi_f$ on $\partial\Omega$, we use the functions v_p and ϕ as local barriers. Indeed, let $p \in \partial\Omega$. Then the functions v_p and ϕ define a modulus of continuity in a neighborhood of p. Since $v_p(p) = \phi(p) = f(x)$, with $p = (x, \epsilon)$, we deduce $u(p) = f(x)$ and this completes the proof of Theorem 9.3.1. □

As a consequence of Theorem 9.3.1, we have [Col90]:

Corollary 9.3.3 *Let $H > 0$ and f be a convex function on $\mathbb{R}$. Let $\Omega \subset \mathbb{R}^2$ be a strip of width $w \le 1/H$. Then there exists a solution of $P(H, \varphi_f)$.*

Proof The proof can be carried out by using nodoids or half cylinders. For example, assume $w < 1/H$. Since f is convex, f satisfies a uniform exterior ρ_R-circle condition for all $R > 0$. Proposition 8.4.1 gives $h_R(H) \in [0, 1/(2H))$ and the width of the strip where there is a solution takes any value in the interval $(0, 1/H)$. If $w = 1/H$, we replace the nodoids $\mathscr{N}_R$ by the half cylinder with equation $z = \sqrt{1/(4H^2) - y^2}$. We may also use pieces of half cylinders for the first part corresponding to width $w < 1/H$. □

The height estimate (9.4) has been obtained thanks to the type of domain, in this case, a strip, and the width $w \leq 2h_R(H)$ allows us to control the height of an H-graph with pieces of H-nodoids.

The second existence theorem in this section concerns the case $\varphi = 0$ in the Dirichlet problem, but now Ω is not a strip.

Theorem 9.3.4 *Let $H > 0$ and let $\Omega \subset \mathbb{R}^2$ be an unbounded domain that satisfies a uniform exterior R-circle condition, $0 < R < \infty$. If Ω is included in a strip of width $w < 2h_R(H)$, then there exists a solution of $P(H, 0)$.*

The solution obtained in Theorem 9.3.4 is unique: [CK91, Hwa88, Maz07a, Mik80].

Proof Consider $\Omega \subset \{(x, y) \in \mathbb{R}^2 : 0 < y < w\}$. Denote by P the plane $z = 0$. We already know that the solutions of (8.1) with zero boundary data are bounded. Consider $\mathscr{N}_R$ as in the proof of Theorem 9.3.1 and let u be a solution of (8.1)–(8.2) for $\varphi = 0$. Without loss of generality, assume that $u > 0$. The same argument used to find the height estimate in Theorem 9.3.1 proves the existence of a constant C depending only on $\partial\Omega$, such that $0 \leq u < C < h_R$.

Let us fix H. Consider the 1-parameter family of H-nodoids $\mathscr{N}_t$. Let $r_1 < R$ so that $h_{r_1} = C$. Position $\mathscr{N}_{r_1}$ in such way that the axis of $\mathscr{N}_{r_1}$ coincides with the z-axis and the waist circle of $\mathscr{N}_{r_1}$ lies in P. Let $\mathscr{N}^{r_1}$ be the part of $\mathscr{N}_{r_1}$ above the plane P.

Move $\mathscr{N}^{r_1}$ down by a small displacement such that the new radius r of the circle obtained by intersecting with the horizontal plane $z = 0$ satisfies $r_1 < r < R$ (and $C < h_r$). Denote by $\mathscr{N}^r$ again this part of the nodoid above P.

Let $\Omega^r \subset \Omega$ be the annulus onto which $\mathscr{N}^r$ vertically projects. The radii of Ω^r are $r < \rho_r$, where the first circle lies in the plane P and the second one in a horizontal plane at height $z = h_r$.

Arguing as in the proof of Theorem 9.3.1, we introduce the class of subfunctions relative to $f = 0$. Now, we have the minimal solution $\phi = 0$ on Ω. Consider $p \in \partial\Omega$ and move $\mathscr{N}^r$ horizontally so that the disk D_r defined by the circle $\partial\mathscr{N}^r \cap P$ satisfies $D_r \subset \mathbb{R}^2 \setminus \Omega$ and $\overline{D_r} \cap \overline{\Omega} = \{p\}$. This is possible because Ω satisfies a uniform exterior R-circle condition and $r < R$. Write $\Omega^p = \Omega^r$ to indicate the dependence on p and denote by v_p the function on Ω^p whose graph is $\mathscr{N}^r$.

Let Σ be the family of functions v_p with $p \in \partial\Omega$. Then we see again that $0 < v_p$ on $\Omega^p \cap \Omega$. Now the steps to follow are analogous. Let F^* be the class of all

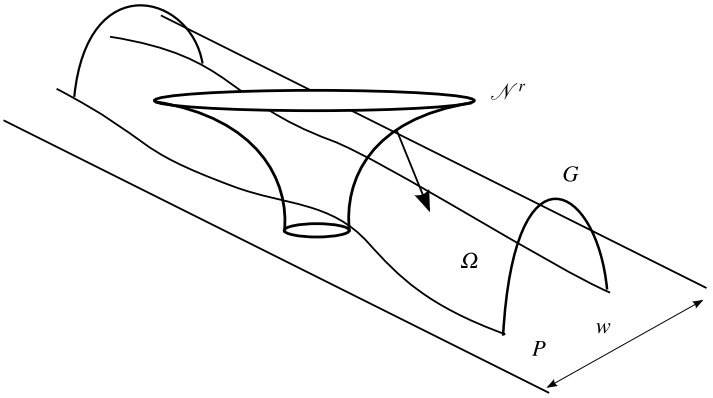

Fig. 9.5 Proof of Theorem 9.3.4. The nodoid $\mathscr{N}^r$ moves down until it reaches $\partial\Omega$ and the graph G lies below $\mathscr{N}^r$. This proves that the solution u satisfies $u = 0$ at $\partial\Omega$

subfunctions v such that $0 \leq v \leq v_p$ in $\Omega^p \cap \Omega$ for all $p \in \partial\Omega$. Recall the graph G of u lies above the plane P. We see $F^* \neq \emptyset$ again because $0 \in F^*$ and

$$u(x, y) = \sup_{v \in F^*} v(x, y) = \sup_{v \in F^*} M_D(v)(x, y)$$

is a solution of (8.1) in Ω.

We employ the functions v_p and 0 in a neighborhood of p for the modulus of continuity. This occurs as follows. We lift $\mathscr{N}^r$ until it does not touch the graph G, and next, we descend it. The tangency principle guarantees the non-existence of interior contact points between $\mathscr{N}^r$ and G before $\mathscr{N}^r$ arrives at the plane P because the orientation of $\mathscr{N}^r$ points down as depicted in Fig. 9.5. Let us remark that the height of G is strictly less than C and $C < h_r$, with h_r the height of $\mathscr{N}^r$. This proves that $\phi < u < v_p$ on $\Omega^p \cap \Omega$. Finally, we use that $v_p(p) = 0 = \phi(p)$. □

Chapter 10
Constant Mean Curvature Surfaces in Hyperbolic Space

This chapter is devoted to the study of compact surfaces with constant mean curvature in hyperbolic space $\mathbb{H}^3$. Although many questions have their origin in Euclidean space, there are some exceptions. One of the first ways in which the geometry of $\mathbb{H}^3$ differs from that of Euclidean space is in the variety of umbilical surfaces. Besides totally geodesic surfaces and spheres, there are more umbilical surfaces, namely, equidistant surfaces and horospheres. As in Euclidean space, there are foliations by totally geodesic surfaces, whose leaves are minimal surfaces, and by spheres, where each leaf has (different) constant mean curvature. However, new foliations appear which are associated with the new umbilical surfaces. So, given $h \in (0, 1)$, there exist foliations of $\mathbb{H}^3$ by equidistant surfaces with the same mean curvature h and if $h = 1$, the foliations are given by horospheres. By means of these foliations, and in combination with the tangency principle, we can sweep $\mathbb{H}^3$ by surfaces, all having the same constant mean curvature, obtaining a certain control of the geometry of an H-surface when there is a relation between the values H and h.

A second feature of $\mathbb{H}^3$ is the existence of cmc surfaces with no equivalence in $\mathbb{R}^3$. We expand on this. The nature of H-surfaces in hyperbolic space depends on the value H of the mean curvature as compared with the number $|c|$, where $c = -1$ is the (constant) sectional curvature of $\mathbb{H}^3$. In a certain sense, surfaces with $|H| = 1$ exhibit similar behaviour as minimal surfaces of $\mathbb{R}^3$. This relation was first studied by R. Bryant [Bry87] and has been successfully continued by many authors. If $|H| > 1$, the surfaces behave similarly to (non-minimal) cmc surfaces of $\mathbb{R}^3$. It is the case $|H| < 1$ that is completely new and which has no counterpart in Euclidean space.

A third difference appears when we consider the simplest boundary curve. Given a circle Γ of radius ρ, there is a family of umbilical surfaces spanning Γ and such that the values of the mean curvature H vary in an interval. First H starts from zero, corresponding with a geodesic surface, next H grows by means of equidistant surfaces until $H = 1$, say, a horosphere. Next, we have spherical caps (pieces of spheres), where the value of the mean curvature increases until it attains maximum of the curvature of the circle, namely, $H = \coth \rho$ (greater than 1). Finally, increasing the radius of spheres, H decreases towards $H = 1$. As in Euclidean space, we

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_10,
© Springer-Verlag Berlin Heidelberg 2013

want to investigate whether the umbilical surfaces are the only possible examples with boundary a circle. In this sense, the same three conjectures of Chap. 1 are considered in hyperbolic ambient space.

We begin the chapter with the description of umbilical surfaces. As in Euclidean space, the tangency principle and the flux formula will give us a certain control and information about the shape of a compact cmc surface of $\mathbb{H}^3$ in terms of the geometry of its boundary. We shall prove that a compact H-surface spanning a circle with $|H| \leq 1$ must be umbilical. Finally, we end the chapter by studying cmc surfaces bounded by a circle under the hypothesis that the surface is a topological disk or stable, as was considered in Chap. 7 for cmc disks of $\mathbb{R}^3$. Some of the results are taken from [Lop99d].

10.1 Basics on Hyperbolic Geometry

In this section we fix the notation and we recall some properties of hyperbolic space $\mathbb{H}^3$. We shall employ the upper half-space model for $\mathbb{H}^3$, that is,

$$\mathbb{R}^3_+ = \left\{(x, y, z) \in \mathbb{R}^3 : z > 0\right\}$$

equipped with the metric

$$\langle \cdot, \cdot \rangle = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

We shall use the affine structure of $\mathbb{R}^3$, we will denote by (x, y) the horizontal coordinates and the z-axis will indicate the vertical direction. Let $e_3 = (0, 0, 1)$.

The hyperbolic space $\mathbb{H}^3$ has a natural compactification $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$, where $\partial_\infty \mathbb{H}^3 = \{z = 0\} \cup \{\infty\}$ is the asymptotic boundary of $\mathbb{H}^3$ and it agrees with the one-point compactification of the xy-plane $P_0 = \{z = 0\}$. Furthermore, $\partial_\infty \mathbb{H}^3$ can be identified with the asymptotic classes of geodesic rays in $\mathbb{H}^3$.

Let $x : M \to \mathbb{R}^3_+$ be an immersion of an orientable surface M. Consider on M the induced metric from the Euclidean space and from the hyperbolic space. In this setting, there exists a relation between the mean curvature of both isometric immersions. A more general result refers to conformal metrics as follows.

Lemma 10.1.1 *Let $(Q, \bar{g})$ be a Riemannian n-dimensional manifold and let f be a smooth positive function on Q. Define the metric $g = f^2 \bar{g}$. Let $x : M \to Q$ be an immersion of an orientable hypersurface. If $\bar{H}$ stands for the mean curvature of $(M, x^*(\bar{g}))$ with respect to an orientation $\bar{N}$, then the mean curvature H of $(M, x^*(g))$ is*

$$H = \frac{\bar{H}}{f} - \frac{1}{f^2} \bar{N}(f) \tag{10.1}$$

computed with the orientation $N = \bar{N}/f$, where $\bar{N}(f)$ is the derivative of f along $\bar{N}$.

Proof If ∇ and $\bar{\nabla}$ stand for the Levi-Civita connections of (Q, g) and $(Q, \bar{g})$, respectively, we have

$$\nabla_X Y = \bar{\nabla}_X Y + \frac{1}{f}\big(X(f)Y + Y(f)X - g(X,Y)\bar{\nabla} f\big) \quad X, Y \in \mathfrak{X}(M), \tag{10.2}$$

where $\bar{\nabla} f$ is the gradient of f for the metric $\bar{g}$. Let u be a principal direction for the isometric immersion $x : (M, x^*(\bar{g})) \to (Q, \bar{g})$, $\bar{\nabla}_u \bar{N} = -\bar{\kappa} u$, $\bar{\kappa} \in \mathbb{R}$. We show that $v = u/f$ is a principal direction for $x : (M, x^*(g)) \to (Q, g)$. Using (10.2),

$$\begin{aligned}\nabla_v N &= \bar{\nabla}_v N + \frac{1}{f}\big(v(f)N + N(f)v\big) = \frac{1}{f^3}\bar{N}(f)u + \frac{1}{f^2}\bar{\nabla}_u \bar{N} \\ &= \frac{1}{f^3}\bar{N}(f)u - \frac{\bar{\kappa}}{f^2}u = \left(\frac{1}{f^2}\bar{N}(f) - \frac{\bar{\kappa}}{f}\right)v.\end{aligned}$$

Moreover, this computation shows that

$$\kappa = \frac{\bar{\kappa}}{f} - \frac{1}{f^2}\bar{N}(f)$$

is a principal curvature, which proves (10.1). □

In the case of hyperbolic space, we choose $Q = \mathbb{R}^3_+$, $\bar{g}$ the Euclidean metric, $g = \langle \cdot, \cdot \rangle$ the hyperbolic metric of $\mathbb{H}^3$ and $f = 1/z$. Let N be a unit normal vector field on $M \subset \mathbb{H}^3$ and consider the orientation $\bar{N}$ on $M \subset \mathbb{R}^3$ given by $\bar{N} = N/z$. The previous computation permits us to establish that the principal curvatures $\kappa_i(p)$ and $\bar{\kappa}_i(p)$ of M are related by

$$\kappa_i(p) = z(p)\bar{\kappa}_i(p) + \bar{N}_3(p) \quad i = 1, 2, \tag{10.3}$$

where $\bar{N}_3 = z \circ \bar{N}$. The relation (10.1) between H and $\bar{H}$ is now

$$H(p) = z(p)\bar{H}(p) + \bar{N}_3(p). \tag{10.4}$$

A first consequence is the classification of the umbilical surfaces of $\mathbb{H}^3$. In the upper half-space model of $\mathbb{H}^3$ and from (10.3), we have $\kappa_1 = \kappa_2$ if and only if $\bar{\kappa}_1 = \bar{\kappa}_2$.

Proposition 10.1.2 *Let $x : M \to \mathbb{H}^3$ be an immersion of a surface M in the upper half-space model of $\mathbb{H}^3$. Then $x : M \to \mathbb{H}^3$ is umbilical if and only if $x : M \to \mathbb{R}^3_+$ is umbilical considering on M the Euclidean metric.*

Therefore, in this model of $\mathbb{H}^3$, umbilical surfaces are the intersection of Euclidean planes and Euclidean spheres with $\mathbb{R}^3_+$. See Fig. 10.1. In what follows, if A is a subset of $\mathbb{R}^3$, we denote by the same symbol A its intersection with $\mathbb{R}^3_+$. Denote by $B(p, r)$ the Euclidean ball centered at p with radius $r > 0$ and $S(p, r) = \partial B(p, r)$. In addition, $\Pi(p, v)$ denotes the Euclidean plane through p and with unit orthogonal vector v.

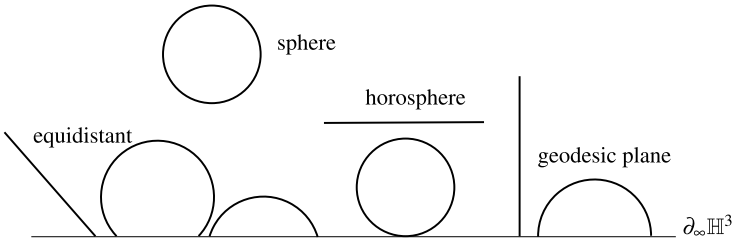

Fig. 10.1 Umbilical surfaces in $\mathbb{H}^3$

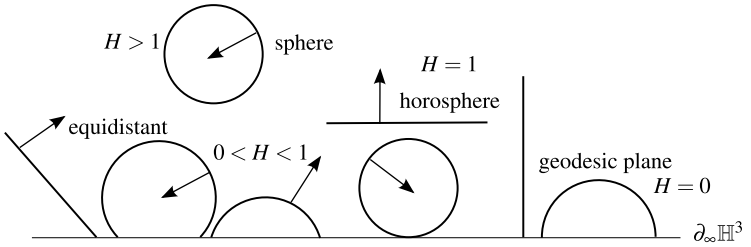

Fig. 10.2 The mean curvature of umbilical surfaces. We indicate the orientations corresponding to non-negative values of the mean curvature

We now compute the mean curvature of the umbilical surfaces of $\mathbb{H}^3$ by using (10.4). Recall that the Gauss curvature of a surface in $\mathbb{H}^3$ is $K = \kappa_1\kappa_2 - 1$. For a plane $\Pi(p_0, v)$, take $\bar{N}(p) = v$. As $\bar{H} = 0$, $H(p) = z(v) = v_3$, the third coordinate of the vector v. In particular, $|H| \leq 1$. For a Euclidean sphere $S(p_0, r)$, consider the inward orientation $\bar{N}(p) = -(p - p_0)/r$. Here $\bar{H} = 1/r$ and $\bar{N}_3(p) = -(z(p) - z(p_0))/r$. Then (10.4) gives

$$H(p) = \frac{z(p)}{r} - \frac{z(p) - z(p_0)}{r} = \frac{z(p_0)}{r}. \tag{10.5}$$

In particular, if the center p_0 lies below P_0, the mean curvature is negative.

We describe the umbilicalsurfaces of $\mathbb{H}^3$ in the upper half-space model. See in Fig. 10.2 the orientations corresponding with $H \geq 0$.

1. *Geodesic plane*. This is an Euclidean hemisphere that orthogonally intersects P_0 and a vertical Euclidean plane. The mean curvature is zero. Intrinsically, the curvature is $K = -1$ and thus it is isometric to the hyperbolic plane $\mathbb{H}^2$.
2. *Equidistant surface*. This is a surface equidistant from a geodesic plane. In our model, it is a tilted Euclidean plane $\Pi(p, v)$, where v is neither vertical nor horizontal, and a Euclidean spherical cap (not a hemisphere) included in $\mathbb{R}^3_+$ and whose boundary is a circle in P_0. The value of H satisfies $0 < |H| < 1$ and K is constant with $K \in (-1, 0)$.
3. *Horosphere*. A horosphere is a 'sphere' whose centre lies at $\partial_\infty \mathbb{H}^3$. In our model, it is a horizontal plane $L(c) = \{(x, y, z) \in \mathbb{R}^3_+ : z = c\}$, $c > 0$, and a Euclidean

sphere of $\mathbb{R}^3_+$ tangent to P_0. The mean curvature H is 1. It is isometric to a Euclidean plane since $K = 0$.

4. *Hyperbolic sphere*. A hyperbolic sphere is the set of equidistant points from a fixed point called its centre. It is a Euclidean sphere included in $\mathbb{R}^3_+$. The mean curvature H satisfies $|H| > 1$. If $\rho > 0$ denotes the hyperbolic radius of the sphere, the mean curvature is $H = \coth \rho$. The curvature K is constant with $K > 1$.

Recall the types of isometries of $\mathbb{H}^3$ viewed in the upper half-space model.

1. *Hyperbolic translation along a geodesic* γ. Every geodesic $\gamma \subset \mathbb{H}^3$ determines a 1-parameter group of hyperbolic translations where the corresponding Killing vector field extends the velocity vector field of γ. If we consider the geodesic $\gamma(t) = (0, 0, t)$, the associated hyperbolic translations are given by

$$\phi_t : (x, y, z) \longmapsto t(x, y, z) \quad t > 0, \tag{10.6}$$

that is, (Euclidean) homotheties from the origin $O \in P_0$. These isometries fix two points of $\partial_\infty \mathbb{H}^3$, namely, $\gamma \cap \partial_\infty \mathbb{H}^3$.
2. *Hyperbolic reflection*. Let P be a geodesic plane represented by a hemisphere of $\mathbb{R}^3_+$ perpendicular to P_0. A hyperbolic reflection with respect to P is Euclidean inversion about P. If P is a vertical Euclidean plane, the corresponding hyperbolic reflection is the Euclidean reflection about P. A hyperbolic reflection fixes pointwise the geodesic plane P.
3. *Parabolic translation*. Thisisometry fixes only one point of $\partial_\infty \mathbb{H}^3$. In our model, it is a Euclidean translation along a fixed horizontal direction.
4. *Hyperbolic rotation*. Consider a geodesic of $\mathbb{H}^3$ given by a vertical line γ. The hyperbolic rotations determined by γ are the Euclidean rotations about γ. These isometries fix γ pointwise.

The relation (10.4) allows us to obtain an expression for the mean curvature in local coordinates of $\mathbb{H}^3$. Let p be a point of a surface M immersed in $\mathbb{R}^3_+$. Around p, we know that M is locally the Euclidean graph on one of the three coordinates planes. If the plane is vertical, then it is a geodesic plane. If the tangent plane is horizontal, we consider the surface locally as a graph on a hemisphere of $\mathbb{R}^3_+$ represented by a geodesic plane of $\mathbb{H}^3$. Thus a surface is locally the graph on a geodesic plane. After a rigid motion of $\mathbb{H}^3$, suppose that M is a graph on the geodesic plane $y = 0$. Let Ω be a domain of the plane $y = 0$, $u \in C^2(\Omega)$ and $M = \text{graph}(u)$. Using (10.4) and (3.3), the mean curvature H of M satisfies

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \frac{2}{z}\left(H + \frac{u_z}{\sqrt{1 + |\nabla u|^2}}\right). \tag{10.7}$$

Here ∇ and div stand for the Euclidean gradient and divergence operators. Again, Eq. (10.7) is of quasilinear elliptic type, such as Eq. (3.3). In this sense, we apply the maximum principle and Theorem 3.2.4 holds in hyperbolic space with the

same statement. Here we need to indicate the differences with the Euclidean case. Equation (10.7) transforms into

$$z\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)-\frac{u_z}{\sqrt{1+|\nabla u|^2}}=2H.$$

We follow the same steps as in Sect. 3.2. The function F is replaced by $F:\mathbb{R}^6\to\mathbb{R}$ given by

$$F(z,x_1,x_2,x_3,x_4,x_5)=z\frac{(1+x_5^2)x_1-2x_4x_5x_2+(1+x_4^2)x_3}{(1+x_4^2+x_5^2)^{3/2}}-\frac{x_5}{\sqrt{1+x_4^2+x_5^2}}.$$

If $H_1\le H_2$, then we begin with

$$0\le F\big(z,r(1),s(1),t(1),p(1),q(1)\big)-F\big(z,r(0),s(0),t(0),p(0),q(0)\big)$$

and the arguments are similar. The quadratic form Q is replaced by zQ, which is elliptic again because $z>0$. We omit the details.

On the other hand, the formulation of the comparisonprinciple is exactly as in Theorem 3.1.5. Indeed, if we repeat to the proof given there, at the point $q=(x_0,z_0)$ we find that

$$u_{xx}(x_0,z_0)+u_{zz}(x_0,z_0)=\frac{2}{z_0}H(x_0,z_0).$$

As a consequence of the comparison principle, we have a control of the mean curvature of a closed surface of $\mathbb{H}^3$.

Theorem 10.1.3 *Let $x:M\to\mathbb{H}^3$ be an immersion of a closed surface M. Then there exists a $q\in M$ such that $|H(q)|>1$. In particular, if M is an H-surface, then $|H|>1$.*

Proof Since M is a compact surface without boundary, there is a ball $B(p,r)\subset\mathbb{R}^3_+$ such that $x(M)\subset B(p,r)$. Move $S(p,r)$ by horizontal translations, which are parabolic translations, until the first contact point q with M. Since $\partial M=\emptyset$, the point q is a common tangent point of M and $S(p,r)$. At q, we orient $S(p,r)$ by the inward orientation N_S which implies that the mean curvature H_S of $S(p,r)$ is positive with $H_S>1$. Consider on M the orientation N such that $N(q)=N_S(q)$. Since $x(M)$ is contained in $B(p,r)$, we have $x(M)\ge S(p,r)$ around q and the comparison principle yields $H(q)\ge H_S>1$. □

In the case $\partial M\neq\emptyset$, we use a comparison argument as above with spheres to control the geometry of a surface whose mean curvature H satisfies $|H|\le 1$ (compare with Corollary 5.3.4).

Proposition 10.1.4 *Let $x : M \to \mathbb{H}^3$ be an immersion of a compact surface M with $\partial M \neq \emptyset$. Assume that the mean curvature function H satisfies $|H| \leq 1$. In the upper half-space model of $\mathbb{H}^3$, we have*

$$x(M) \subset \bigcap \{B(p,r) \subset \mathbb{R}^3_+ : x(\partial M) \subset B(p,r)\}.$$

Proof Consider the family of horospheres of type $L(c)$. Starting with the plane $z = 0$, we increase c until $L(c)$ first touches M at $c = c_0 > 0$. The mean curvature of $L(c_0)$ is 1 with respect to the upward orientation (see Fig. 10.2). If there is a tangent (interior or boundary) point $q \in M \cap L(c_0)$ and orienting M so its Gauss map agrees with that of $L(c_0)$ at q, the comparison principle says that $1 \leq H(q)$. As $|H| \leq 1$, we have $H(q) = 1$. Now, the tangency principle implies $x(M) \subset L(c_0)$ and $x(M)$ is the domain of $L(c_0)$ bounded by $x(\partial M)$. In this case, the result is trivial.

Therefore, we assume that $x(M)$ is not included in a horizontal plane of $\mathbb{R}^3_+$. In view of the above argument with horospheres of type $L(c)$, denote by $c_0 > 0$ the first time that a horizontal plane $L(c)$ touches M, which occurs at a boundary point by the previous argument. Then $x(int(M)) \subset \{(x, y, z) \in \mathbb{R}^3 : z > c_0\}$. Let us take $B(p,r) \subset \mathbb{R}^3_+$ and $x(\partial M) \subset B(p,r)$. The lowest point $\bar{p}$ of $S(p,r)$ satisfies $z(\bar{p}) < c_0$ and thus $\bar{p} \notin M$. Consider the 1-parameter family of Euclidean spheres $\{S(p_t, r + t) : t \geq 0\}$, where $p_t = p + te_3$. All these spheres have $\bar{p}$ as the lowest point. Since M is compact, for t sufficiently large, we have $x(M) \subset B(p_t, r + t)$. Now we perform the reverse process by decreasing the parameter t. As $x(\partial M) \subset B(p,r)$, it is not possible that for some $t > 0$, $S(p_t, r + t)$ touches $x(\partial M)$. On the other hand, the mean curvature of $S(p_t, r + t)$ is greater than 1 with respect to the inward orientation. This means that in this process and by the comparison principle, the existence of a contact point between $S(p_t, r + t)$ and $x(int(M))$ for $t > 0$ is forbidden. This forces t to take the initial value $t = 0$ and we return to the initial position of $B(p,r)$, showing that $x(M) \subset B(p,r)$. □

The key ingredient in the above proof is that the mean curvature of $S(p,r) \subset \mathbb{R}^3_+$ is greater than 1. A similar argument can be carried out by replacing $S(p,r)$ by geodesic planes and for minimal surfaces. The next result is the hyperbolic version of Theorem 3.3.1.

Proposition 10.1.5 *Let $x : M \to \mathbb{H}^3$ be an immersion of a compact surface M with $\partial M \neq \emptyset$. If $H = 0$, then $x(M)$ lies in the convex hull of $x(\partial M)$. In particular, if $x(\partial M)$ lies in a geodesic plane P, then $x(M) \subset P$.*

Here, by convex hullwe mean the following. A geodesic plane separates $\mathbb{H}^3$ into two half-spaces. Then the convex hull of $x(\partial M)$ is the intersection of the closures of all such half-spaces containing $x(\partial M)$.

In the case that the boundary lies in another umbilical surface, it is possible to obtain a similar result assuming a certain relation between the mean curvature of the surface and of the umbilical surface. The proof is based in the following result that was mentioned in the introduction of this chapter.

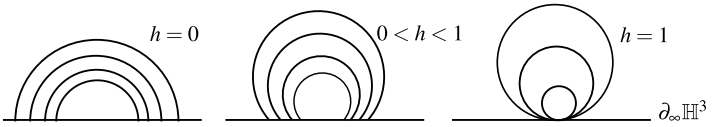

Fig. 10.3 Foliations of $\mathbb{H}^3$ by geodesic planes, equidistant surfaces and horospheres of type $S(he_3, 1)$

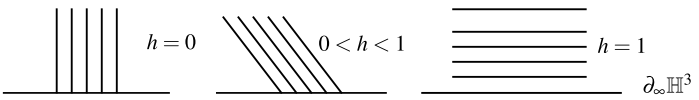

Fig. 10.4 Foliations of $\mathbb{H}^3$ by geodesic planes, equidistant surfaces and horospheres of type $\Pi(p, v)$

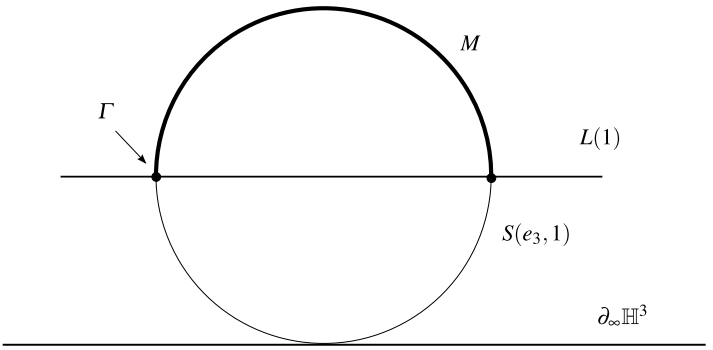

Fig. 10.5 The surface M is embedded with $|H| = 1$. The boundary ∂M lies in the horosphere $L(1)$ but M is not included in $L(1)$

Lemma 10.1.6 *For each $h \in [0, 1]$, there exists a foliation of $\mathbb{H}^3$ by umbilical h-surfaces.*

Proof In the upper half-space model of $\mathbb{H}^3$, let $\phi_t : \mathbb{H}^3 \to \mathbb{H}^3$ be the hyperbolic translations $\phi_t(p) = tp$, $t > 0$. Consider $P = S(he_3, 1)$. The foliation that we are looking for is $\{P_t = \phi_t(P) : t \in (0, \infty)\}$. See Figs. 10.3 and 10.4. In the latter figure, the umbilical h-surfaces are described in the upper half-space model by Euclidean planes. □

Therefore, fixing $h \in [0, 1]$ and a given cmc surface, we can sweep the space $\mathbb{H}^3$ comparing the surface with umbilical surfaces with the *same* constant mean curvature h. Notice that in case that $|H| = h \neq 0$, an H-surface M with boundary ∂M included in an umbilical surface P with mean curvature h may not be included in P as occurs if $P = L(1)$ and $M = S(e_3, 1) \cap \{(x, y, z) \in \mathbb{R}^3_+ : z \geq 1\}$: see Fig. 10.5. In this setting, we need the following definition.

Definition 10.1.7 Let P be an umbilical surface of $\mathbb{H}^3$ such that P is not a geodesic plane. The mean convex sideof P is the connected component of $\mathbb{H}^3 \setminus P$ towards which the mean curvature vector of P points.

Therefore, the mean convex side is the connected component of $\mathbb{H}^3 \setminus P$ towards which N points when the mean curvature of P is positive. This concept appeared for Euclidean space in Chap. 3.

Corollary 10.1.8 *Let P be a horosphere or an equidistant surface with mean curvature $h > 0$ and let $\Gamma \subset P$ be a simple closed curve. Let $x : M \to \mathbb{H}^3$ be an immersion of a compact surface M spanning Γ and whose mean curvature function H satisfies $|H| \leq h$. Then either $x(int(M))$ lies in the mean convex side of P or $x(M)$ is included in P and $|H| = h$.*

Proof The proofs in both cases of umbilical surfaces are similar, so we shall only give the proof in the case when P is a horosphere. Suppose that $P = L(c)$ for some $c > 0$. Since M is compact, let $t > 0$ be sufficiently small so that $M \subset \{(x, y, z) \in \mathbb{R}^3_+ : z > t\}$. Consider the horosphere $L(t)$ and increase t until the first time $t = t_0$ such that $L(t_0)$ intersects M at a point $p \in M \cap L(t_0)$. Assume that p is an interior or a boundary point where both surfaces are tangent. We know that the unit normal vector in $L(t_0)$ for $h = 1$ is $N'(p_0) = e_3$. We orient M by the Gauss map N so that $N(p) = e_3$ and thus, $M \geq L(t_0)$ around p. By the comparison principle, $H(p) \geq 1$ and since $|H| \leq 1$, we conclude that $H(p) = 1$. Using the tangency principle, M and $L(t_0)$ agree in a neighborhood of p and by connectedness, $M \subset L(t_0)$. Since $\partial M \subset L(c)$, it follows that $t_0 = c$.

The other possibility is that $L(t_0)$ intersects M at a boundary point, where M and $L(t_0)$ are not tangent. Since $\partial M \subset L(c)$, we have $t_0 = c$, proving $x(int(M)) \subset \{(x, y, z) \in \mathbb{R}^3_+ : z > c\}$. □

In Fig. 10.5, M is part of the horosphere $S(e_3, 1)$ and its boundary lies in the horosphere $L(1)$. In this example, the surface M lies in the mean convex side of $L(1)$, which agrees with the result.

10.2 The Flux Formula

The simplest boundary curve is a circle. In hyperbolic space $\mathbb{H}^3$, we call a *spherical cap* the domain of a hyperbolic sphere bounded by a circle. Besides geodesic disks and spherical caps, there are more umbilical surfaces spanning circles and this highlights the difference between hyperbolic and Euclidean space. Consider a circle Γ of radius $\rho > 0$ included in a geodesic plane P. After an isometry of $\mathbb{H}^3$, assume that $\Gamma \subset L(1)$ and is centered on the z-axis. From the Euclidean viewpoint, P is a hemisphere with $\partial P \subset P_0$ and Γ is a circle of Euclidean radius $r = \sinh(\rho)$ that bounds a domain $\Omega \subset P$. The range of H for the umbilical H-surfaces spanning Γ is the

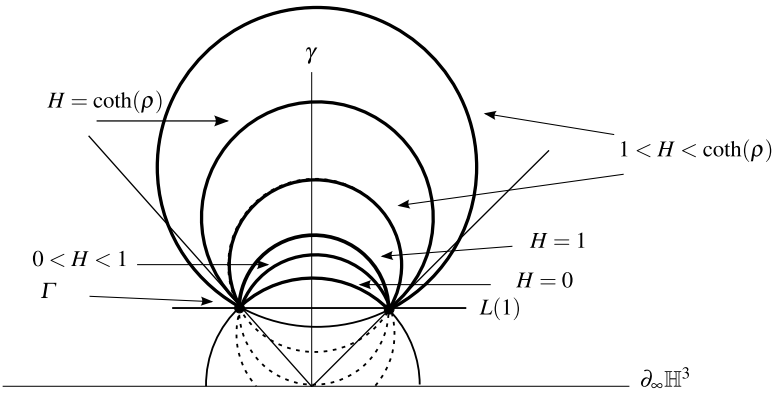

Fig. 10.6 Umbilical surfaces spanning a circle Γ

following. See Fig. 10.6. Consider Euclidean spheres centered on the z-axis containing Γ, more precisely, $S(te_3, R_t)$, $t \geq 0$, where $R_t^2 = (t-1)^2 + r^2$. Then Γ defines in $S(te_3, R_t)$ two subdomains with common boundary Γ. If $S(te_3, R_t)$ is a hyperbolic sphere ($t > \cosh(\rho)^2/2$), both subdomains are compact, indeed, two spherical caps; assuming the contrary, only one domain is a compact surface. Starting with $\overline{\Omega}$, H takes the value $H = 0$. Recall that $P = S(O, \sqrt{1+r^2}) = S(O, \cosh(\rho))$. As we increase t, we have equidistant surfaces, where the value of H increases from 0 to $H = 1$, which is attained by the horosphere $S(te_3, R_t)$ for the value $t = r^2/2$. Next, take the spheres $S(te_3, R_t)$ for $t > r^2/2$. By (10.5), the mean curvature of the spherical cap of $S(te_3, R_t)$ spanning Γ is

$$H = \frac{t}{R_t} = \frac{t}{\sqrt{(t-1)^2 + \sinh(\rho)^2}}.$$

The value of H increases until it attains a maximum at $t = \cosh(\rho)^2$ with $H = \coth(\rho)$. Here the surface is $S(\cosh(\rho)^2 e_3, \sinh(\rho)\cosh(\rho)) \cap \{(x, y, z) \in \mathbb{R}^3_+ : z \geq 1\}$, which is a spherical cap orthogonally intersecting P along Γ. This surface plays the same role as a hemisphere in Euclidean space because its mean curvature is the maximum among all values of mean curvature of spherical caps spanning a given circle Γ. Finally, increasing $t \to \infty$, we produce a family of spherical caps where the value of H decreases towards $H = 1$.

We call attention to the fact that these umbilical surfaces are the only compact cmc surfaces that are rotationally symmetric (invariant by hyperbolic rotations) spanning a circle.

As in Euclidean space, we shall obtain restrictions on the possible values of H among all H-surfaces spanning a given closed curve. This is obtained using a version of the flux formula in hyperbolic space [KKMS92, NE96, NR95]. Let M be an immersed H-surface in $\mathbb{H}^3$ spanning a closed curve Γ of length L. Denote by M' a compact surface such that $M \cup M'$ bounds an oriented domain of $\mathbb{H}^3$ and represent the induced orientations on M and M' by N and $N_{M'}$, respectively. Let Y be a

Killing vectorfield of $\mathbb{H}^3$. As in Corollary 5.1.4, we obtain

$$2H\int_{M'}\langle N_{M'},Y\rangle\,dM'=\int_{\partial M}\langle \nu,Y\rangle\,ds, \tag{10.8}$$

where ν is the inward unit conormal vector of M along ∂M. This means that the possible values of H are restricted by the geometry of Γ. Indeed, the integral $\int_{M'}\langle N_{M'},Y\rangle\,dM'$ depends only on Γ since M' is a compact surface spanning Γ. Choose M' so that this integral is not zero and denote it by $A(\Gamma,M')$. Thus we have

$$|H|\leq \sup_{\Gamma}|Y|\frac{L}{2|A(\Gamma,M')|} \tag{10.9}$$

and this number depends only on Γ. In contrast with the inequality (5.7) in Euclidean space, the expression (10.9) is not so amenable in general. The estimate (10.9) can be improved by replacing the denominator with the supremum of all $|A(\Gamma,M')|$ (compare with (5.8)).

We look at a particular example. Suppose Γ is a simple closed curve included in a geodesic plane P. Denote by $\Omega\subset P$ the domain bounded by Γ and let $a(\Omega)$ be its area. After an isometry, assume that $P=S(O,r)$. Consider $Y(p)=p$, which is a Killing vector field of $\mathbb{H}^3$ in the upper half-space model. Let us take $M'=\Omega$. Here $N_\Omega(p)=z(p)p/r$. Let

$$\Theta=\min_{p\in\partial\Omega}z(p).$$

On Ω we find that

$$\langle N_\Omega,Y\rangle=\frac{r}{z}\geq 1.$$

Along $\Gamma=\partial\Omega$, $\langle \nu,Y\rangle\leq|p|=r/z(p)$. Then Eq. (10.8) implies

$$2|H|a(\Omega)\leq\int_\Gamma\langle\nu,Y\rangle\,ds\leq\int_\Gamma\frac{r}{z}\,ds\leq\frac{rL}{\Theta},$$

where L is the length of Γ. Thus

$$|H|\leq\frac{rL}{2\Theta\,a(\Omega)}. \tag{10.10}$$

In the particular case where Γ is a circle, we have the following theorem of B. Nelli and H. Rosenberg [NR95]:

Theorem 10.2.1 *Let Γ be a circle of radius $\rho>0$ and let M be a compact H-surface immersed in $\mathbb{H}^3$ spanning Γ. Then*

$$|H|\leq\coth(\rho). \tag{10.11}$$

Moreover, if $H=0$, M is included in a geodesic plane and if $|H|=\coth(\rho)$, the surface is umbilical.

Proof There is no loss of generality in assuming that Γ is included in $L(1)$ and centered on the z-axis. Then $\Gamma \subset P = S(O, \cosh(\rho))$. If $H = 0$, we know by Proposition 10.1.5 that $M \subset P$. Assume now that $H \neq 0$. With the previous notation, we have

$$L = 2\pi \sinh(\rho), \qquad a(\Omega) = \pi \sinh(\rho)^2, \qquad r = \cosh(\rho), \qquad \Theta = 1.$$

Then (10.10) yields

$$|H| \leq \coth(\rho).$$

Suppose now $|H| = \coth(\rho)$. Then all the above inequalities in the flux formula are equalities, in particular, ν is proportional to Y, i.e., $\nu(q) = z(q)q$ for $q \in \Gamma$. This means that N is orthogonal to P along ∂M. If $\alpha = \alpha(s)$ is a parametrization of Γ by the arc length, then $\sigma(\alpha'(s), \nu(s)) = 0$ and ∂M is a line of curvature. Taking into account that the Euclidean radius of Γ is $r = \sinh(\rho)$, (10.2) yields

$$\begin{aligned}\sigma\left(\alpha', \alpha'\right) &= \left\langle N, \nabla_{\alpha'}\alpha' \right\rangle = \left\langle \bar{N}, \bar{\nabla}_{\alpha'}\alpha' \right\rangle + \langle \bar{N}, \bar{\nabla} z \rangle \\ &= \frac{1}{r\cosh(\rho)} + \frac{r}{\sqrt{1+r^2}} = \coth(\rho).\end{aligned}$$

This shows that the boundary ∂M is formed by umbilical points, in particular, they are not isolated. Then Theorem 2.2.2 implies that M must be umbilical. □

The last part of the statement of this result is the hyperbolic version of Brito-Earp's theorem for $\mathbb{R}^3$, see Corollary 5.1.11.

10.3 Embedded cmc Surfaces in $\mathbb{H}^3$

We study the relationship between the symmetries of a given closed curve Γ and those of a compact cmc surface spanning Γ. If the surface is embedded, the Alexandrov reflection methodpresents the same analogies as in Euclidean space.

Theorem 10.3.1 *Let Γ be a simple closed curve contained in a geodesic plane P. Assume that Γ is symmetric with respect to a hyperbolic reflection about a geodesic plane Π orthogonal to P and let $L = \Pi \cap P$. Assume that each connected component of $\Gamma \setminus (\Gamma \cap L)$ is a graph on L. Let M be a compact embedded cmc surface in $\mathbb{H}^3$ with boundary Γ. If M lies on one side of P then M is symmetric with respect to the plane Π.*

We say that each connected component of $\Gamma \setminus (\Gamma \cap L)$ is a graph on L if for each point $q \in L$, the geodesic in P starting at q and perpendicular to L intersects at most once each one of the above components.

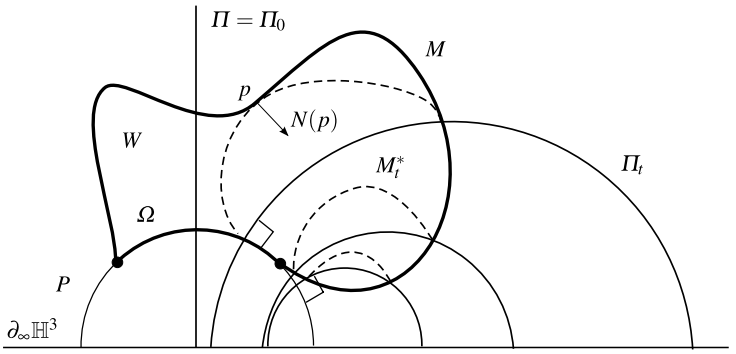

Fig. 10.7 The Alexandrov reflection method in $\mathbb{H}^3$ about geodesic planes Π_t

Proof Let $\Omega \subset P$ be the domain bounded by Γ. Since M lies on one side of P, $M \cup \Omega$ encloses a 3-domain W. Consider on M the Gauss map N pointing towards W. Let γ be a geodesic in P perpendicular to L and let Π_t be the 1-parameter family of geodesic planes orthogonal to γ that foliates $\mathbb{H}^3$ with $\Pi_0 = \Pi$. The parameter $t \in \mathbb{R}$ indicates the signed distance from Π_t to the point $\gamma \cap \Pi_0$: see Fig. 10.7.

Hyperbolic reflections about the geodesic planes Π_t are inversions about Π_t. As Π_t is orthogonal to P for all t, the reflection of P about Π_t coincides, as a subset of $\mathbb{R}^3_+$, with P. Now the Alexandrov reflection method works as in Theorem 4.1.1. Using a similar notation, for $t > 0$, M_t^+ (resp. M_t^-) is the part of M inside (resp. outside) the Euclidean half-ball determined by Π_t and the plane P_0 and M_t^* is the inversion of M_t^+ about Π_t. Starting with sufficiently large t, decrease t until Π_t first touches M at $t = t_0 > 0$. A standard argument as in the proof of Theorem 4.1.1, using the tangency principle, shows that the first contact point p must occur at $t_1 \geq 0$. If $t_1 > 0$ and by the hypothesis that each connected component of $\Gamma \setminus (\Gamma \cap L)$ is a graph on L, then $p \in int(M_{t_1}^*) \cap int(M_{t_1}^-)$ or $p \in (\partial M_{t_1}^* \cap \partial M_{t_1}^-) \setminus \Gamma$. In both cases, the tangency principle implies that Π_{t_1} is a plane of symmetry of M, in particular, of Γ, a contradiction.

Thus $t_1 = 0$. Moreover $M_0^* \subset \overline{W}$. If there is a tangent point between M_0^* and M_0^-, then Π is a plane of symmetry of M, proving the result. Assuming the contrary, we proceed by reflections with planes Π_t coming from $t = -\infty$, obtaining a tangent point before arriving at $t = 0$, and subsequently, a plane of symmetry, which is a contradiction. □

Corollary 10.3.2 *Let $M \subset \mathbb{H}^3$ be a compact embedded cmc surface spanning a circle Γ and let P be the geodesic plane containing Γ. If M lies on one side of P, then M is an umbilical surface.*

In the case when $\partial M = \emptyset$, the Alexandrov process works as in Theorem 10.3.1.

Corollary 10.3.3 *The only closed embedded cmc surface in hyperbolic space is the round sphere.*

Proof We need to indicate the proof because, in contrast to the Euclidean setting, there is no notion of parallelism in $\mathbb{H}^3$. We proceed by reflections with respect to geodesic planes represented by vertical planes. Similarly as in Euclidean space, this proves that the surface M is invariant by hyperbolic rotations about a geodesic of $\mathbb{H}^3$ represented by a vertical line. Now, fixing $p_0 \in P_0$, we carry out the reflection method using hyperbolic reflections about $\{S(p_0, r) : r > 0\}$. There exists an $r(p_0) > 0$ such that M is invariant under the hyperbolic reflection about $S(p_0, r(p_0))$. Since this holds for any $p_0 \in P_0$, M must be a round sphere. □

We have studied the case where the boundary Γ lies in a geodesic plane. We may assume that Γ is included in another type of umbilical surface, say, a horosphere or an equidistant surface. In both cases, P separates the space $\mathbb{H}^3$ into two connected components. If M is a surface in $\mathbb{H}^3$ with boundary in P, we say that M lies on one side of P if $int(M)$ lies in one of the two components of $\mathbb{H}^3 \setminus P$.

Theorem 10.3.4 *Let Γ be a simple closed curve contained in a horosphere or in an equidistant surface P. Assume that Γ is symmetric with respect to a hyperbolic reflection about a geodesic plane Π orthogonal to P and that each connected component of $\Gamma \setminus (\Gamma \cap L)$ is a graph on $L = \Pi \cap P$. If M is a compact embedded cmc surface in $\mathbb{H}^3$ with boundary Γ and M lies on one side of P, then M is symmetric with respect to the plane Π. If Γ is a circle, then the surface is umbilical.*

Let us give a precise definition of a graph on P. Each connected component A of $\Gamma \setminus (\Gamma \cap L)$ is a graph on $L \subset P$, where P is a horosphere (resp. equidistant surface) if for each point $q \in L$, the horocycle (resp. equidistant line) in P starting at q and perpendicular to L intersects at most one point of A.

Proof Assume first that P is a horosphere. Without loss of generality, suppose P is $L(1)$ and Π is the plane $y = 0$. Let γ be a horocycle in $L(1)$ perpendicular to $\Pi \cap L(1)$, which means that γ is a horizontal line with equations $\{x = c, z = 1\}$, $c \in \mathbb{R}$. Let Π_t be the 1-parameter family of geodesic planes $y = t$, $t \in \mathbb{R}$, with $\Pi_0 = \Pi$, which are all orthogonal to γ. Observe that the hyperbolic reflection about Π_t leaves invariant, as subsets, the horosphere $L(1)$ and each of the two connected components of $\mathbb{H}^3 \setminus L(1)$ as depicted in Fig. 10.8. From the Euclidean viewpoint, the reflection process works exactly as in Theorem 4.1.1, obtaining the result.

Suppose now that P is an equidistant surface. After a rigid motion, consider P the sphere $S(\lambda e_3, 1)$, $-1 < \lambda < 0$, and Π the vertical plane $y = 0$. Let γ be an equidistant line included in P and orthogonal to Π and let Π_t, $t \in \mathbb{R}$, be the family of hemispheres orthogonally intersecting γ with $\Pi_0 = \Pi$. The parameter t is the signed distance between the point $\gamma \cap \Pi_t$ and Π. See Fig. 10.9. As subsets of $\mathbb{H}^3$, P and the two connected components of $\mathbb{H}^3 \setminus P$ are invariant by hyperbolic reflections about Π_t (Euclidean inversions). We proceed with large positive values of t and next, let $t \to 0$. We apply the Alexandrov method as usual denoting by M_t^+ (resp. M_t^-) the part of M included in (resp. outside) the Euclidean half-ball determined

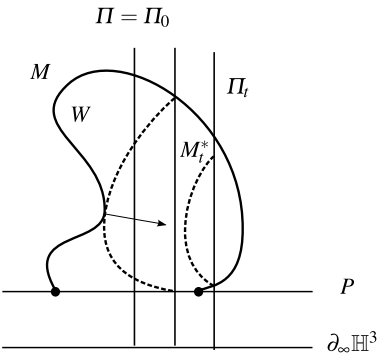

Fig. 10.8 The Alexandrov reflection method when the umbilical surface P is a horosphere

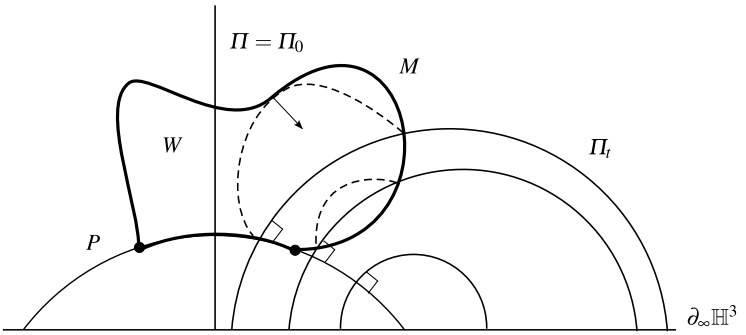

Fig. 10.9 The Alexandrov reflection method when the umbilical surface P is an equidistant surface

by $\Pi_t \cup P_0$ and M_t^* the reflection of M_t^+ about Π_t. Again, the tangency principle proves that there is a tangent point between $M_{t_1}^*$ and $M_{t_1}^-$ if and only if $t_1 = 0$.

If Γ is a circle, take Π to be a geodesic plane perpendicular to P through the center of Γ. The above result shows that M is invariant under reflection with respect to Π. Repeating this process for all such geodesic planes Π, we conclude that M is a rotational surface with respect to the geodesic of $\mathbb{H}^3$ orthogonal to P through the center of Γ. Hence M is an umbilical surface spanning Γ. □

As in Euclidean space, we can replace the condition of prescribing the boundary curve by the assumption that the contact angle between the surface and the support surface is constant along its boundary. The arguments are similar to those of Theorem 4.1.16 so we only state the result [Lop06b].

Corollary 10.3.5 *Let M be a compact embedded cmc surface in $\mathbb{H}^3$ whose boundary lies in P, where P is a geodesic plane, an equidistant surface or a horosphere. Assume that the angle between M and P is constant along ∂M. If M lies on one side of P, then M is an umbilical surface and ∂M is a circle.*

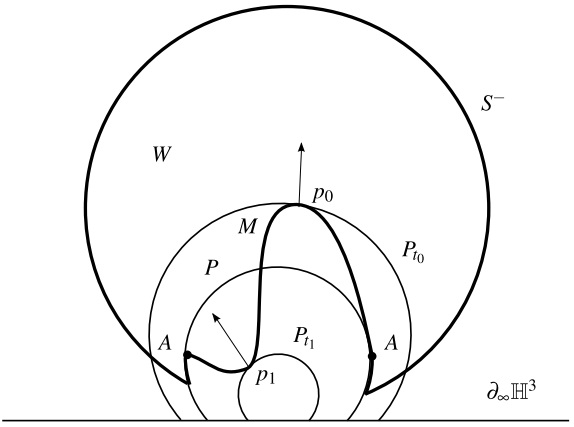

Fig. 10.10 Proof of Theorem 10.3.6 when the umbilical surface P is an equidistant surface

All the previous results show that when the boundary curve Γ lies in a geodesic plane, an equidistant surface or a horosphere P, the condition that the surface lies on one side of P allows the use of the Alexandrov method. A first result that establishes this property is motivated by the extension of Koiso's theorem (Theorem 3.1.11).

Theorem 10.3.6 *Consider a simple closed curve Γ included in P, where P is a geodesic plane, an equidistant surface or a horosphere with mean curvature $h \geq 0$. Let $\Omega \subset P$ be the domain bounded by Γ. Consider a compact embedded surface $M \subset \mathbb{H}^3$ spanning Γ whose mean curvature function H satisfies $|H| \neq h$. If $M \cap (P \setminus \overline{\Omega}) = \emptyset$, then M lies on one side of P. If, in addition, H is constant and Γ is a circle, then M is an umbilical surface.*

Proof Since the proof is similar in the three cases of umbilical support, we only treat the case when P is an equidistant surface. Suppose that P is the sphere $S(he_3, 1)$. We proceed by contradiction, assuming that M has points on both sides of P. We adopt the notation of the preceding lemma. Let I^+ and I^- denote the two sides of $\mathbb{H}^3$ determined by P:

$$I^- = \bigcup_{t>1} P_t, \qquad I^+ = \bigcup_{0<t<1} P_t.$$

As $h \geq 0$, the orientation on P points to I^+. Consider the function $f : M \to (0, \infty)$ that gives the leaf of the foliation: $f(p) = t$ if $p \in P_t$. Since M is compact, there exist interior points $p_0, p_1 \in M$ such that f achieves its maximum and minimum, respectively. Let $t_0 = f(p_0)$ and $t_1 = f(p_1)$. By hypothesis, suppose $t_1 \leq 1 < t_0$ (the case $t_1 < 1 \leq t_0$ is similar).

Let $R > 0$ be sufficiently large such that for some $\delta > 0$, the surface M lies in the Euclidean ball $B((R + \delta)e_3, R)$ and let $S = \partial B((R + \delta)e_3, R)$. Consider the non-smooth closed surface $T = S^- \cup A \cup M$, where $S^- = S \cap I^-$ and $A \subset P$ is the annulus bounded by ∂S^- and Γ as depicted in Fig. 10.10. Then T encloses a

bounded 3-domain $W \subset \mathbb{H}^3$. We orient T by the unit normal vector field pointing towards W and let N be the induced orientation on M.

Place P_{t_1} at the point p_1 and orient P_{t_1} in such a way that its Gauss map at p_1 is $N(p_1)$. Since $N(p_1)$ points towards W, the orientation on P_{t_1} corresponds with the mean curvature $-h$. As $M \geq P_{t_1}$ around p_1, the comparison principle yields $H(p_1) \geq -h$. Observe that this holds even in the case $t_1 = 1$ because $p_1 \in int(M)$. By hypothesis, $|H| \neq h$, which implies $H(p_1) > -h$. Now compare P_{t_0} and M around p_0. Since $N(p_0)$ points to W, if we orient P_{t_0} so that it coincides with $N(p_0)$ at p_0, then $P_{t_0} \geq M$ around p_0. The comparison principle now says $-h \geq H(p_0)$ and as $|H| \neq h$, we have $-h > H(p_0)$. Thus the function H attains on M values greater than and less than $-h$. By continuity, $H = -h$ at a point of M, a contradiction. □

Finally, we point out that Theorem 5.4.1 holds in the hyperbolic setting assuming that the boundary lies in a geodesic plane. The proof follows the same arguments ([NR95]; the corresponding result in $\mathbb{S}^n$ can be found in [Lir02a]).

Theorem 10.3.7 *Let P be a geodesic plane of $\mathbb{H}^3$ and let M be a compact embedded cmc surface in $\mathbb{H}^3$ whose boundary is a convex curve included in P. If M is transverse to P along ∂M, then M lies on one side of P. In the particular case that ∂M is a circle, M is an umbilical surface.*

10.4 Graphs in Hyperbolic Space

In this section we work with the concept of a graph in $\mathbb{H}^3$. There are two different notions of graph defined on a domain of a geodesic plane.

Definition 10.4.1 Consider a domain Ω of a geodesic plane P and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

1. Geodesic graph. This is the set of points $\{p_q : q \in \Omega\}$, where p_q is the point in the geodesic γ_q orthogonal to P at q at oriented distance $u(q)$ from q. See Fig. 10.11.
2. Killing graph. Let $p_0 \in P$ and γ_{p_0} be the geodesic through p_0 and perpendicular to P. Consider the hyperbolic translations along γ_{p_0} and the corresponding Killing vector field of $\mathbb{H}^3$ associated to these translations. For each $q \in \Omega$, let O_q be the orbit of q and denote by p_q the point of O_q at oriented distance $u(q)$ from q. The Killing graph of u with respect to p_0 is defined as the set $\{p_q : q \in \Omega\}$. See Fig. 10.12.

Both concepts extend easily to define curves of $\mathbb{H}^3$ that are geodesic or Killing graphs over a curve of a geodesic plane.

In the upper half-space model, let P be a geodesic plane represented by a hemisphere $S = S(O, r)$ and let $p_0 = (0, 0, r)$. Then a Killing graph on $\Omega \subset S$ with

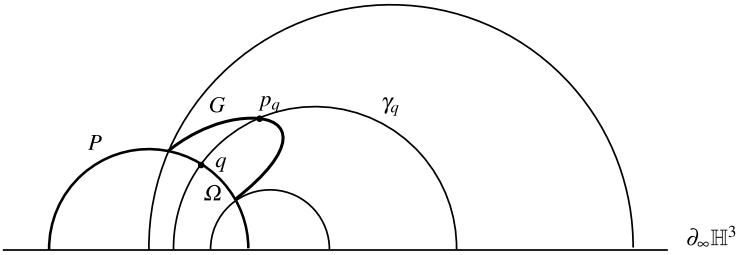

Fig. 10.11 A geodesic graph G in $\mathbb{H}^3$ on the domain $\Omega \subset P$

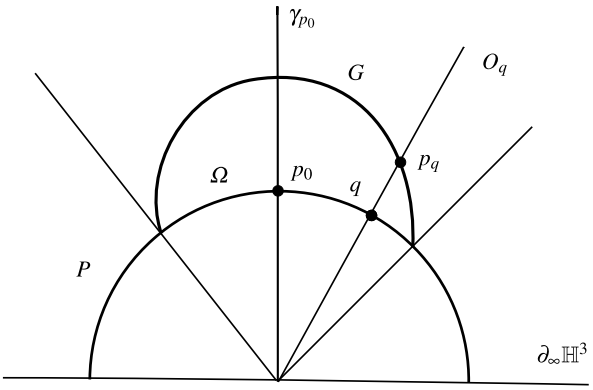

Fig. 10.12 A Killing graph G in $\mathbb{H}^3$ on the domain $\Omega \subset P$ with respect to p_0

respect to p_0 is a radial graph on Ω from the Euclidean viewpoint. For Killing graphs, there is uniqueness of H-graphs.

Proposition 10.4.2 *Let Ω be a bounded domain of a geodesic plane P and $p_0 \in P$. Let $\Gamma \subset \mathbb{H}^3$ be a curve that is a Killing graph on $\partial\Omega$ with respect to p_0. Then there exists at most one Killing H-graph on Ω with respect to p_0 and with boundary Γ.*

Proof Assume that $P = S(O, 1)$ and $p_0 = (0, 0, 1)$. Since M is a radial graph on a domain of $S(O, 1)$, the function $\langle N(p), p\rangle$ does not vanish on M, where N is the Gauss map of M. Let M_1 and M_2 be two Killing H-graphs with $\partial M_1 = \partial M_2 = \Gamma$ and suppose that the orientations N_i satisfy $\langle N_i(p), p\rangle > 0$, $p \in M_i$, $i = 1, 2$. We shall assume that both graphs have the same orientation.

Consider the hyperbolic translations $\phi_t(p) = tp$, $t > 0$. See Fig. 10.13. For sufficiently large $\phi_t(M_1)$ does not intersect M_2. Letting $t \to 1$, we continue until the first intersection point p_1 with M_2. This will occur at a time $t_1 \geq 1$. If this contact point is a tangent (interior or boundary) point, and because $N_1(p_1) = N_2(p_1)$, the tangency principle implies that both surfaces agree on an open subset around p_1. By connectedness, $\phi_{t_1}(M_1) = M_2$. As $\partial\phi_{t_1}(M_1) = t_1\partial M_1$, it follows that $t_1 = 1$ and $M_1 = M_2$, proving the result.

Assuming the contrary, we continue until $t = 1$, the original position of M_1, and no tangent points exist between the two surfaces. In particular, along Γ the slope

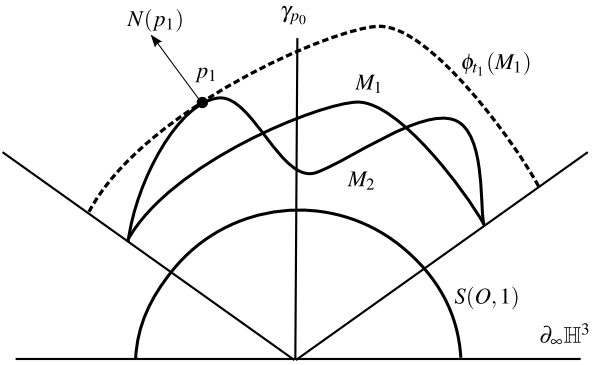

Fig. 10.13 Proposition 10.4.2: uniqueness of Killing graphs

of M_2 is strictly less than that of M_1, that is,

$$\langle \nu_2(q), q \rangle < \langle \nu_1(q), q \rangle, \quad q \in \Gamma, \tag{10.12}$$

where ν_i denotes the unit inward conormal vector of M_i along Γ. Now, we proceed in the same way with the surface M_2. By (10.12), M_2 lies strictly below M_1 along the common boundary and thus, in the process of translating M_2, there must exist $t_2 > 1$ such that $\phi_t(M_2) \cap M_1 = \emptyset$ for $t > t_2$ and $\phi_{t_2}(M_2) \cap M_1 \neq \emptyset$. Then necessarily there is a common tangent interior point $p_2 \in M_1 \cap \phi_{t_2}(M_2)$. The tangency principle implies that $\phi_{t_2}(M_2) = M_1$ in an open set around p_2 and by connectedness, $\phi_{t_2}(M_2) = M_1$. However $\partial \phi_{t_2}(M_2) = t_2 \Gamma \neq \Gamma$, obtaining a contradiction. □

There are two notions of cylinders in $\mathbb{H}^3$.

Definition 10.4.3 Let Ω be a domain of a geodesic plane P.

1. Geodesic cylinder. This is the set of all geodesics orthogonal to P through all points of Ω.
2. Killing cylinder. Given $p_0 \in P$ and γ_{p_0} the orthogonal geodesic to P at p_0, the Killing cylinder on Ω with respect to p_0 is the set of all orbits O_q, $q \in \Omega$, produced by the Killing vector field associated to γ_{p_0}. We denote it by $K(\Omega, p_0)$.

If Ω is a round disk of radius $\rho > 0$ in a geodesic plane, a Killing cylinder, $p_0 = (0, 0, 1)$, on Ω is written as

$$K(\rho, p_0) = \left\{ (x, y, z) \in \mathbb{R}^3_+ : x^2 + y^2 \leq \left(\sinh(\rho)^2\right) z^2 \right\}. \tag{10.13}$$

The mean curvature H of $\partial K(\rho, p_0)$ is constant, with

$$H = \frac{1}{2}\left(\tanh(\rho) + \coth(\rho)\right) > 1$$

with respect to the inward orientation.

Proposition 10.4.4 *Let Ω be a bounded domain of a geodesic plane P and $p_0 \in P$. Let $\Gamma \subset \partial K(\Omega, p_0)$ be a closed curve that is a Killing graph on $\partial\Omega$ with respect to p_0. If M is a compact embedded cmc surface spanning Γ and $int(M) \subset K(\Omega, p_0)$, then M is a Killing graph on Ω with respect to p_0.*

Proof As in Proposition 10.4.2, assume that $P = S(O, 1)$ and $p_0 = (0, 0, 1)$. We apply the Alexandrov reflection method by using hyperbolic reflections about the 1-parameter family $P_t = \phi_t(P)$, $t \in (0, \infty)$, i.e., inversions about $S(O, t)$. Note that these reflections leave invariant as subsets of $\mathbb{R}^3_+$ both $\partial K(\Omega, p_0)$ and $K(\Omega, p_0)$. Since M is embedded and $\Gamma \subset \partial K(\Omega, p_0)$, the surface M separates $K(\Omega, p_0)$ into two connected components and let W be the domain that intersects $B(O, t)$ for all t sufficiently close to $t = 0$. Denote by $M_t^+ = M \cap (\mathbb{R}^3_+ \setminus B(O, t))$, $M_t^- = M \cap B(O, t)$ and M_t^* the reflection of M_t^+ about P_t.

Let the planes P_t approach from $t = \infty$ until the first intersection point with M. Next, let $t \searrow 0$ and reflect M_t^+ about P_t until M_t^* first touches M_t^- at $t = t_1$. If this number t_1 does not exist, then M is a Killing graph and the result is proved. If not, we continue until the first position that $M_{t_1}^* \not\subset \overline{W}$ and thus, there is a point $p \in M_{t_1}^+$ such that $p^* \in M_{t_1}^-$. Since Γ is a Killing graph, we have $p, p^* \notin \Gamma$. Thus, either $p^* \in int(M_{t_1}^*) \cap int(M_{t_1}^-)$ or $p^* \in (\partial M_{t_1}^* \cap \partial M_{t_1}^-) \setminus \Gamma$. In both cases, $M_{t_1}^*$ and $M_{t_1}^-$ are tangent at p^* and one surface lies above the another around p^*. The tangency principle shows that P_{t_1} is a plane of symmetry which is not possible because Γ is a Killing graph on $\partial\Omega$ with respect to p_0. □

If we only assume that the surface is immersed, we derive a similar result as in Theorem 5.2.1.

Theorem 10.4.5 *Let Ω be a bounded domain of a geodesic plane P and $p_0 \in P$. Assume that there exists a Killing H-graph G on Ω with respect to p_0 and with boundary $\partial\Omega$. Let M be a compact surface and let $x : M \to \mathbb{H}^3$ be an immersion of constant mean curvature H spanning $\partial\Omega$. If $x(int(M)) \subset K(\Omega, p_0)$, then, either $x(M) = G$ or $x(M) = G^*$, the hyperbolic reflection of G with respect to P.*

Proof The case $H = 0$ is trivial by Proposition 10.1.5. Let $H \neq 0$. Assume that the geodesic plane is $S(O, 1)$ and $p_0 = (0, 0, 1)$. We know that G lies on one side of P (Theorem 10.3.6) and there is no loss of generality in assuming that G lies over P. We orient G by the Gauss map N_G so that $\langle N_G(p), p\rangle < 0$, $p \in G$. Comparing with the geodesic planes $\phi_t(P)$ in the farthest point of G from the origin O, we deduce that the mean curvature of G is positive. Move G up by hyperbolic translations ϕ_t and then down until the first contact point with M. Note that $\phi_t(\partial\Omega) \subset \partial K(\Omega, p_0)$. An analogous process can be carried out with G^* and translations coming from $t = 0$. Using the tangency principle in the same way as in Theorem 5.2.1, we obtain that either $x(M) = G$, or $x(M) = G^*$ or $x(M)$ lies in the 3-domain bounded by $G \cup G^*$ and $x(M)$ is not tangent to G and G^* along $\partial\Omega$ as depicted in Fig. 10.14. We prove that the latter case is not possible.

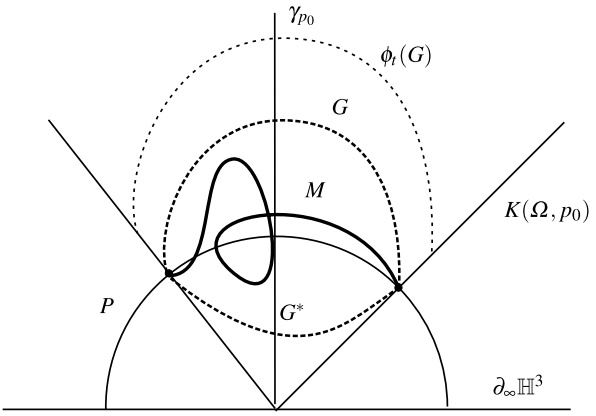

Fig. 10.14 Theorem 10.4.5: uniqueness on Killing cylinders

Apply the flux formula (10.8) with the Killing vector field $Y(p) = p$. Let ν and ν_G be the inward unit conormal vectors of M and G along $\partial\Omega$, respectively. The last condition on $\partial\Omega$ says

$$\left|\langle \nu(p), p\rangle\right| < -\langle \nu_G(p), p\rangle, \quad p \in \partial\Omega. \tag{10.14}$$

Let $M' = \Omega$ in (10.8). The flux formula for M and G gives

$$\left|\int_{\partial\Omega} \langle \nu, p\rangle\, ds\right| = 2H\int_\Omega \langle p, p\rangle\, d\Omega = \left|\int_{\partial\Omega} \langle \nu_G, p\rangle\, ds\right|.$$

Thus

$$\left|\int_{\partial\Omega} \langle \nu, p\rangle\, ds\right| = \left|\int_{\partial\Omega} \langle \nu_G, p\rangle\, ds\right|$$

and inequality (10.14) yields a contradiction. □

A result analogous to Corollary 4.2.2 can be obtained in hyperbolic space. Since the arguments are similar, we omit the proof.

Theorem 10.4.6 *Let Ω be a simply connected bounded domain of a geodesic plane P, $p_0 \in P$. Let Γ be a simple closed curve that is a Killing graph on $\partial\Omega$ with respect to p_0. Let M be a compact embedded cmc surface in $\mathbb{R}^3$ with boundary Γ such that $int(M)$ does not intersect one of the two connected components of $\partial K(\Omega, p_0) \setminus \Gamma$. If a neighborhood of ∂M in M is a Killing graph on Ω, then M is a Killing graph on Ω with respect to p_0.*

If the mean curvature of the surface is constant, we can relate the notions of geodesic graph and Killing graph.

Proposition 10.4.7 *Let Ω be a bounded domain of a geodesic plane P. If M is a compact cmc geodesic graph on Ω spanning $\partial\Omega$, then M is a Killing graph on Ω.*

Proof If $H = 0$, $M = \overline{\Omega}$ by Proposition 10.1.5, which is a Killing graph. If $H \neq 0$, then M lies on one side of P (Theorem 10.3.6). Assume that $P = S(O, 1)$ and $int(M)$ lies outside $B(O, 1)$. We consider the family of planes P_t as in Proposition 10.4.4 and apply the same technique of Alexandrov, where W is the domain bounded by $M \cup (P \setminus \overline{\Omega})$ that contains Ω. Approaching from $t = \infty$, if there is no contact point between M_t^* and M_t^- until we arrive at $t = 1$, then M is a Killing graph.

Assuming the contrary, there is a contact point p^* between $M_{t_1}^*$ and $M_{t_1}^-$ for some $t_1 > 1$ with $p \in M_{t_1}^+$. If $p^* \notin \partial\Omega$, the tangency principle proves that $M_{t_1}^* = M_{t_1}^-$ and P_{t_1} is a plane of symmetry, which is a contradiction because $\partial\Omega$ lies below P_{t_1}. We discard the last possibility, namely, $p^* \in \partial\Omega$. Since M is a geodesic graph on Ω and M_t^+ is a Killing graph on Ω for all $t > t_1$, this case only occurs if M is perpendicular to P at p^*. In particular, T_pM is orthogonal to P_{t_1}. This contradicts that $M_{t_1}^+$ is a Euclidean radial graph. □

For the converse, and as a consequence of the uniqueness of Killing cmc graphs given in Proposition 10.4.2, we have:

Corollary 10.4.8 *Let Ω be a bounded domain of a geodesic plane P and $p_0 \in P$. Let M be a Killing H-graph on Ω with respect to p_0 and let G be a geodesic H-graph on Ω. If $\partial M = \partial G = \partial\Omega$, then, up to a hyperbolic reflection about P, $M = G$.*

Finally, we point out another notion of graph in the hyperbolic context, namely, a parabolic graph. Consider a 1-parameter subgroup $\{\psi_t : t \in \mathbb{R}\}$ of isometries of $\mathbb{H}^3$ of parabolic type, that is, ψ_t are parabolic translations whose orbits are horocycles having a common point at infinity. Let P be a geodesic plane orthogonal to the orbits. For a domain $\Omega \subset P$ and a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, the *parabolic graph* of u is defined by $\{\psi_{u(q)}(q) : q \in \overline{\Omega}\}$. In the case that P is a vertical plane, a parabolic graph on $\Omega \subset P$ is a Euclidean graph on Ω because the horocycles orthogonal to P are horizontal lines orthogonal to P. We have [Hin02]:

Proposition 10.4.9 *Let Ω be a bounded domain of a geodesic plane P. Consider a compact cmc surface $M \subset \mathbb{H}^3$ with $\partial M = \partial\Omega$. If M is a geodesic graph on Ω, then M is a parabolic graph on Ω.*

Proof Consider the upper half-space model of $\mathbb{H}^3$ and without loss of generality, assume that P is the vertical plane with equation $y = 0$. If $H = 0$ we know by Proposition 10.1.5 that $M = \overline{\Omega}$, proving the result. Assume that $H \neq 0$. As the surface M does not intersect $P \setminus \overline{\Omega}$, Theorem 10.3.6 proves that M lies on one side of P. Suppose that $int(M)$ is included in the half-space $\{(x, y, z) \in \mathbb{R}^3_+ : y > 0\}$ and let $T = M \cup (\Omega \times (-\infty, 0)e_2)$ with $e_2 = (0, 1, 0)$. Then T separates $\mathbb{H}^3$ into two connected components and denote by W the component containing Ω.

We employ the Alexandrov reflection method using hyperbolic reflections about the geodesic planes P_t, $y = t$. These hyperbolic reflections are here Euclidean reflections with respect to P_t. Approaching from $t = \infty$, we apply the Alexandrov

technique as in Theorem 4.2.1, which here is formally identical. If $M_t^* \subset \overline{W}$ for all $t \geq 0$, then M is a parabolic graph, proving the result. Assuming the contrary, there exists a $t_1 > 0$, the first time where the property $M_t^* \subset \overline{W}$ fails. In such case there are two possibilities:

1. There exists a $p^* \in (M_{t_1}^* \cap M_{t_1}^-) \setminus \Gamma$ with $p \in M_{t_1}^+$. Then p^* is an (interior or boundary) tangent point of both surfaces and $M_{t_1}^* \geq M_{t_1}^-$ around p (with respect to the inward orientations). The tangency principle implies that P_{t_1} is a symmetry plane of M, which is impossible since the boundary of M lies on one side of P_{t_1}.
2. There is a point $p \in M_{t_1}^+$ such that $p^* \in \partial M$. In particular, $p \neq p^*$ and $p^* \notin P_{t_1}$. Then the horizontal segment between p and p^* lies in W. Since M is a geodesic graph, the tangent plane $T_{p^*}M$ is perpendicular to P. This implies that the same occurs for p and T_pM is orthogonal to P. However, this contradicts that $t_1 > 0$ and that during the reflection process, the surfaces M_t^+ for $t \geq t_1$ are Euclidean graphs on P_{t_1}. □

10.5 The Case $|H| \leq 1$

In this section we study compact H-surfaces in $\mathbb{H}^3$ whose mean curvature satisfies $|H| \leq 1$. In view of Proposition 10.1.5 and Corollary 10.1.8, we have a certain control of the shape of H-surface M whose boundary lies in P, where P is a geodesic plane, an equidistant surface or a horosphere and $|H| \leq h$, h the mean curvature of P. In the particular case that the boundary is a circle, we have [BE98a, BE98b, Lop99d]:

Theorem 10.5.1 *Let $H \in \mathbb{R}$ with $|H| \leq 1$. The only compact H-surfaces immersed in $\mathbb{H}^3$ with circular boundary are the umbilical surfaces.*

Proof Let M be an immersed H-surface with $\partial M = \Gamma$. Without loss of generality, suppose the circle Γ lies in $L(1)$, is centered on the z-axis and is included in the geodesic plane $P = S(O, r)$. Denote by $\Omega \subset P$ the domain bounded by Γ. If $H = 0$, we know by Proposition 10.1.5 that $M = \overline{\Omega} \subset P$. Now assume that $0 < |H| \leq 1$. Let us take $h = |H|$ and let Q and Q^* be the two equidistant surfaces or the two horospheres of constant mean curvature h containing Γ. In fact, Q^* is the hyperbolic reflection of Q with respect to P. By Corollary 10.1.8, M lies in the mean convex sides of Q and Q^*, in particular, M is included in the 3-domain $W \subset \mathbb{R}^3_+$ determined by $Q \cup Q^*$: see Figs. 10.15 and 10.16.

Denote by G (resp. G^*) the compact part of Q (resp. Q^*) whose boundary is Γ. By the geometry of the circle $\Gamma \subset P$, the three surfaces M, G and G^* are included in the Killing cylinder $K(\Omega, p_0)$ where $p_0 = (0, 0, r)$. As G and G^* are Killing graphs on Ω with respect to p_0, Theorem 10.4.5 yields $M = G$ or $M = G^*$, proving that M is umbilical. □

It is worth highlighting the importance of Theorem 10.5.1 when compared with the other results of this book because it makes no assumptions on the topology,

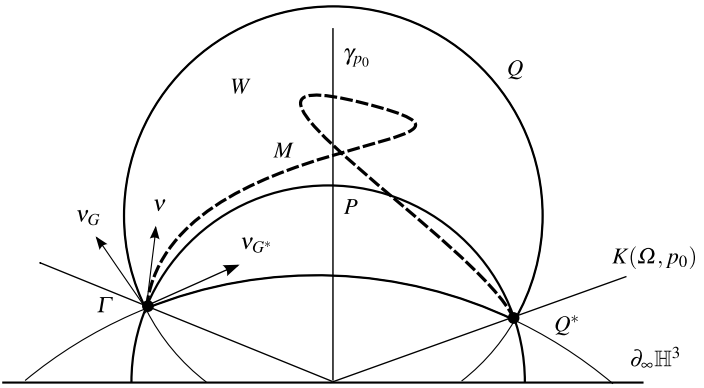

Fig. 10.15 Proof of Theorem 10.5.1. Here $|H| < 1$ and Q is an equidistant surface

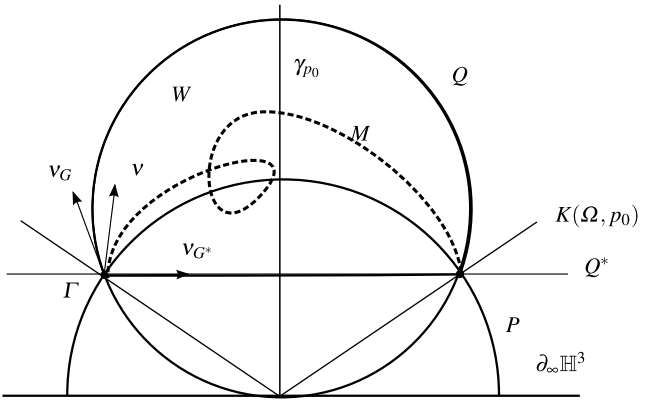

Fig. 10.16 Proof of Theorem 10.5.1. Here $|H| = 1$ and Q is a horosphere

embeddedness or stability of the surface. This is in contrast to what happens in Euclidean space, where there is no counterpart of the family of H-surfaces with $|H| < 1$.

We continue with the case $|H| \leq 1$ now assuming that the boundary of the surface lies in a horosphere. Let Γ be a simple closed curve included in a horosphere $L(c)$ and let $\Omega \subset L(c)$ be the domain bounded by Γ. If $|H| \leq 1$ we know by Corollary 10.1.8 that a compact H-surface $x : M \to \mathbb{H}^3$ with boundary Γ lies in the mean convex side of $L(c)$ given by $x(int(M)) \subset \{(x, y, z) \in \mathbb{R}^3_+ : z > c\}$. Assume now that M is embedded. Then $M \cup \Omega$ bounds a domain W of $\mathbb{H}^3$. Consider on M the orientation corresponding to $H \geq 0$. We say that N points inside (resp. outside) if N points to W (resp. to $\mathbb{R}^3_+ \setminus \overline{W}$). Notice that if $H = 1$, then $M = \overline{\Omega}$ or $int(M) \subset \{(x, y, z) \in \mathbb{R}^3_+ : z > c\}$ with N pointing inside.

We have a new uniqueness result.

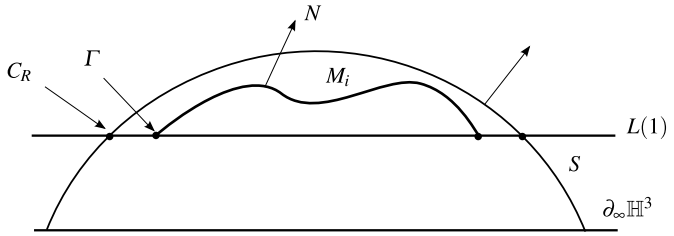

Fig. 10.17 The equidistant surface S in the proof of Theorem 10.5.2

Theorem 10.5.2 *Let $H \in \mathbb{R}$ with $0 \leq H < 1$. Let Γ be a simple closed strictly convex curve included in a horosphere and consider two compact embedded H-surfaces M_1 and M_2 in $\mathbb{H}^3$ spanning Γ. If the orientations on M_i point outside, then $M_1 = M_2$.*

Proof If Γ is a circle, then M is umbilical by Theorem 10.5.1 and the result is immediate. Assume that Γ is not a circle. Suppose that the horosphere containing Γ is $L(1)$. After a horizontal translation (a parabolic translation of $\mathbb{H}^3$), assume that Ω intersects the z-axis.

We consider a disk $D_R \subset L(1)$ of radius $R > 0$ and centered on the z-axis such that $\Gamma \subset D_R$. Let $C_R = \partial D_R$. Since Γ is convex and $L(1)$ is isometric to the Euclidean plane $\mathbb{R}^2$, Γ is also convex in the Euclidean sense. Then we use the rolling property of convex plane curves. Let R be sufficiently large so that C_R can roll along Γ with the property $\Gamma \subset \overline{D_R}$ and touching each point of Γ.

Suppose $H > 0$. Consider the equidistant surface $S = S(\lambda e_3, -\lambda/H)$, $\lambda < 0$, such that $C_R = \partial S^+$, where $S^+ = S \cap \{(x, y, z) \in \mathbb{R}^3_+ : z \geq 1\}$. As $\lambda < 0$ and by (10.5), the mean curvature of S is H for the orientation pointing outside $B(\lambda e_3, -\lambda/H)$. See Fig. 10.17. Consider the hyperbolic translations ϕ_t with respect to the origin O. Let $t > 1$ be sufficiently large so that M_1 and M_2 are included in $\phi_t(B(\lambda e_3, -\lambda/H))$. Next move $\phi_t(S)$ with $t \searrow 1$. It is not possible that for some $t > 1$, $\phi_t(S)$ has a tangent point with M_i because M_i and S have the same mean curvature for the orientations pointing outside. Thus, the value t arrives at $t = 1$ and S returns to its original position. In this situation, M_1 and M_2 are included in the domain bounded by $S^+ \cup D_R$.

We move S^+ by horizontal translations until it reaches M_i. By the tangency principle, there cannot exist an interior or boundary tangent point between M_i and any displacement of S because in such case, an open set of M would be included in S: since S is umbilical, this would say that M is also umbilical by Theorem 2.2.2, but as $\partial M_i \subset L(c)$, ∂M_i is a circle, a contradiction.

By the choice of the radius R, we move S horizontally touching tangentially each point of Γ and by the geometry of S^+, we have $M_i \subset \overline{\Omega} \times \mathbb{R}$. Let $p_0 \in \Omega$ and let L be the geodesic orthogonal to $L(c)$ at p_0. Consider the hyperbolic translations from the point $L \cap P_0$. A similar argument as in the proof of Theorem 10.4.5, together with the property $M_i \subset \overline{\Omega} \times \mathbb{R}$, proves that $M_1 = M_2$.

Suppose now $H = 0$. This case is similar by replacing S by a geodesic plane. Here the hypothesis that the Gauss map points outside is unnecessary because the mean curvature $H = 0$ does not change by reversing the orientations. □

10.6 Surfaces with Circular Boundary

If $|H| \leq 1$, we have seen in Theorem 10.5.1 that the only compact immersed H-surfaces with circular boundary are the umbilical ones. Thus the remaining case to consider is $|H| > 1$. As in Euclidean space $\mathbb{R}^3$, we have:

Conjecture 1 *An H-disk immersed in $\mathbb{H}^3$ with circular boundary and $|H| > 1$ is umbilical.*

Conjecture 2 *A compact embedded H-surface in $\mathbb{H}^3$ with circular boundary and $|H| > 1$ is umbilical.*

Conjecture 3 *A compact stable H-surface in $\mathbb{H}^3$ with circular boundary and $|H| > 1$ is umbilical.*

The next result is similar to Theorem 5.3.1.

Theorem 10.6.1 *Let $H \in \mathbb{R}$ with $|H| > 1$. Let Γ be a circle and let $x : M \to \mathbb{H}^3$ be an immersion of a compact surface M spanning Γ and with constant mean curvature H. If $x(M)$ is included in a closed hyperbolic ball of radius* $\operatorname{arctanh}(1/|H|)$, *then $x(M)$ is a spherical cap.*

Proof Consider the upper half-space model of $\mathbb{H}^3$. By an isometry, assume $\Gamma \subset L(1)$ and that its center lies on the z-axis. Let S be the boundary of the hyperbolic ball of the assumption. If there is a tangent point between $x(M)$ and S, and because the mean curvature of S is $|H|$ with the inward orientation, the tangency principle implies that $x(M) \subset S$ and this shows that $x(M)$ is a spherical cap. Otherwise, let us move S by any horizontal translation and by homotheties from any point of $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$. Again, if there is a tangent point between $x(M)$ and one of the above displacements of S, the tangency principle proves that $x(M)$ is a spherical cap again, proving the result.

Assuming the contrary, there would be a small spherical cap G spanning Γ and with mean curvature H such that $x(M)$ lies in the domain of $\mathbb{H}^3$ bounded by $G \cup G^*$. Here G^* denotes the hyperbolic reflection of G about the geodesic plane P containing Γ. If $\Omega \subset P$ is the round disk bounded by Γ, the surfaces G and G^* are Killing geodesic graphs on Ω with respect to a point p_0, the intersection of P with the geodesic $\gamma(t) = (0, 0, t)$. Since M, G and G^* are included in $K(\Omega, p_0)$, Theorem 10.4.5 concludes the proof. □

Theorem 10.6.1 extends by assuming that $x(M)$ is included in a solid Killing cylinder of appropriate radius. As the argument is similar to Theorem 5.3.5, we only state the result.

Theorem 10.6.2 *Let $H \in \mathbb{R}$ with $|H| > 1$. Let Γ be a circle and let $x : M \to \mathbb{H}^3$ be an immersion of a compact surface M spanning Γ and with constant mean curvature H. If $x(M)$ is included in a solid Killing cylinder of radius* $\operatorname{arctanh}(1/|H|)$, *then $x(M)$ is a spherical cap.*

We finish this section by proving two similar results that appeared in Chap. 7. Assume $H \in \mathbb{R}$ with $|H| > 1$. The isoperimetric inequality of Barbosa-Do Carmoof Lemma 7.1.1 gives a relation between the length L of the boundary Γ of an immersed disk M in $\mathbb{H}^3$ and its area A. Since $K \leq H^2 - 1$, take $c = H^2 - 1$ and we have

$$L^2 \geq 4\pi A - \big(H^2 - 1\big)A^2.$$

Furthermore, equality holds if and only if M is an open set of a hyperbolic sphere. Assume that Γ is a circle of radius $\rho > 0$. Then $L = 2\pi \sinh(\rho)$ and consequently, the area of M satisfies

$$\big(H^2 - 1\big)A^2 - 4\pi A + 4\pi^2 \sinh(\rho)^2 \geq 0.$$

From this inequality, we find that

$$A \leq A_- = \frac{2\pi}{H^2 - 1}\big(1 - \sqrt{\cosh(\rho)^2 - H^2 \sinh(\rho)^2}\,\big)$$

or

$$A \geq A_+ = \frac{2\pi}{H^2 - 1}\big(1 + \sqrt{\cosh(\rho)^2 - H^2 \sinh(\rho)^2}\,\big).$$

The numbers A_- and A_+ correspond to the areas of the two spherical H-caps bounded by Γ and satisfy $0 < A_- \leq A_+$. Note that $\cosh(\rho)^2 - H^2 \sinh(\rho)^2 \geq 0$ by (10.11). Now we are in position to establish the following result.

Theorem 10.6.3 *Let $H \in \mathbb{R}$ with $|H| > 1$. Consider an immersed H-disk M in $\mathbb{H}^3$ spanning a circle Γ and with area A. If $A \leq A_+$, then M is a spherical cap.*

Proof The proof follows the same steps as in Theorem 7.1.2. We only highlight the differences. After an isometry, suppose that $\Gamma \subset L(1)$ is a circle of radius ρ. We know that if $A = A_+$ the surface M is umbilical, more precisely M is the large spherical H-cap bounded by Γ. If not, $A \leq A_-$. In this case, as $K \leq H^2 - 1$, the Gauss-Bonnet theorem says

$$2\pi = \int_M K\, dM + \int_{\partial M} k_g(s)\, ds \leq A_-\big(H^2 - 1\big) + \int_{\partial M} k_g(s)\, ds.$$

Then

$$2\pi\sqrt{\cosh(\rho)^2 - H^2\sinh(\rho)^2} \leq \int_{\partial M} k_g(s)\,ds. \tag{10.15}$$

Squaring (10.15) and using Schwarz's inequality, we get

$$4\pi^2\big(\cosh(\rho)^2 - H^2\sinh(\rho)^2\big) \leq 2\pi\sinh(\rho)\int_{\partial M} k_g(s)^2\,ds. \tag{10.16}$$

If α is an arc length parametrization of Γ, using that Γ is a circle of radius ρ, the geodesic curvature κ_g is

$$k_g = \pm\frac{\sqrt{\cosh(\rho)^2 - \langle \nu, \alpha\rangle^2}}{\sinh(\rho)},$$

where ν denotes the inner unit conormal vector of M along Γ. If we substitute this identity into (10.16), we have

$$\begin{aligned} 4\pi^2\big(\cosh(\rho)^2 - H^2\sinh(\rho)^2\big) &\leq 2\pi\sinh(\rho)\int_{\partial M}\frac{\cosh(\rho)^2 - \langle \nu, \alpha\rangle^2}{\sinh(\rho)^2}\,ds \\ &= 4\pi^2\cosh(\rho)^2 - \frac{2\pi}{\sinh(\rho)}\int_{\partial M}\langle \nu, \alpha\rangle^2\,ds. \end{aligned}$$

Then

$$\int_{\partial M}\langle \nu, \alpha\rangle^2\,ds \leq 2\pi H^2\sinh(\rho)^3. \tag{10.17}$$

On the other hand, take the Killing vector field $Y(p) = p$. By the virtue of the flux formula (10.8), we have

$$\big(2\pi|H|\sinh(\rho)^2\big)^2 = \left(\int_{\partial M}\langle \nu, \alpha\rangle\,ds\right)^2 \leq 2\pi\sinh(\rho)\int_{\partial M}\langle \nu, \alpha\rangle^2\,ds,$$

that is,

$$2\pi H^2\sinh(\rho)^3 \leq \int_{\partial M}\langle \nu, \alpha\rangle^2\,ds.$$

From this inequality and (10.17), we get identities in all inequalities. In particular, $A = A_-$ and the surface is umbilical. □

The last result is analogous to the result established in Theorem 7.2.3.

Theorem 10.6.4 *The only stable cmc disks in hyperbolic space $\mathbb{H}^3$ with circular boundary are umbilical surfaces.*

Proof Consider the Minkowski model of $\mathbb{H}^3$. Let $\mathbb{L}^4$ be the four-dimensional Lorentz-Minkowski space, that is, the vector space $\mathbb{R}^4$ with the Lorentz metric:

$$\langle \cdot, \cdot \rangle = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 - (dx_4)^2,$$

where (x_1, x_2, x_3, x_4) are the usual coordinates of $\mathbb{R}^4$. Then $\mathbb{H}^3$ is isometric to the hyperquadric

$$\mathbb{H}^3 = \left\{ x \in \mathbb{L}^4 : \langle x, x \rangle = -1, x_4 > 0 \right\}$$

with the induced metric from $\mathbb{L}^4$. In this model, the umbilical surfaces of $\mathbb{H}^3$ are the intersection of the umbilical surfaces of $\mathbb{L}^4$ with the hyperquadric $\mathbb{H}^3$ (see Sect. 11.1). Assume that Γ is the circle

$$\Gamma = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{H}^3 : x_1^2 + x_2^2 = r^2, x_3 = 0, x_4^2 = 1 + r^2 \right\}.$$

Consider a conformal immersion $x : D \to \mathbb{H}^3$ of constant mean curvature, where D is the unit disk of $\mathbb{R}^2$. Let $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$. Consider the Killing vector field on $\mathbb{H}^3$ defined by $Y(p) = x \times e_3 \times e_4$. The vector field Y corresponds to the 1-parameter subgroup of rotations of $\mathbb{R}^4$ around the plane passing through the origin and generated by e_3 and e_4. Define the function ψ as

$$\psi = \langle e_3 \times e_4 \times x, N \rangle.$$

As in Lemma 7.2.1, the function ψ satisfies

$$\int_D \psi \, dD = 0, \quad \psi = 0 \text{ along } \partial D.$$

In view of (2.27) and (2.28), the Laplacian $\Delta\psi$ is

$$\Delta\psi + \left(|\sigma|^2 - 2\right)\psi = 0.$$

The Jacobi operator in $\mathbb{H}^3$ is $L = \Delta + |\sigma|^2 - 2$. With the same notation as Theorem 7.2.3, the computation of the normal derivative of ψ along ∂D is

$$\partial_n \psi = \left\langle e_3 \times e_4 \times dx(\mathbf{n}), N \right\rangle + \left\langle e_3 \times e_3 \times x, dN(\mathbf{n}) \right\rangle.$$

As $N \times dx(\mathbf{n}) \times x = dx(\mathbf{t})$ on ∂D, $e_3 \times e_4 \times dx(\mathbf{n})$ and $e_3 \times e_4 \times x$ are both collinear with $dx(\mathbf{t})$. It follows that

$$\partial_n \psi = \pm r \sigma(\mathbf{t}, \mathbf{n}).$$

The remainder of the proof proceeds as in Theorem 7.2.3. □

Chapter 11
The Dirichlet Problem in Hyperbolic Space

We consider the Dirichlet problem for the cmc equation in hyperbolic space $\mathbb{H}^3$. Due to the type of umbilical surfaces in $\mathbb{H}^3$ as well as the different notions of graphs, there is a variety of problems of Dirichlet type. We may consider Killing graphs on a geodesic plane, such as those introduced in Sect. 10.4. Results of existence appear in [ALR12, DR05, GE05, Lir02b, Nit02, Zha05]. If the domain is unbounded, we may use the Perron method as in Chap. 9 (see [ET99, ET00]). For parabolic graphs, see [AM09, BE98a]. For other methods, such as geometric measure theory or a variational approach, see [AR97, Cus09, SS09].

In this chapter we study geodesic graphs defined in a domain Ω of a horosphere, a geodesic plane and an equidistant surface. In order to describe the techniques, we consider the Dirichlet problem when Ω is a bounded domain and the boundary curve is $\partial\Omega$. As in Euclidean space, we shall prove the existence of such graphs provided there is a certain relation between H and the value of the mean curvature $H_{\partial\Omega}$ of $\partial\Omega$ as submanifold of Ω. The main results are taken from [Lop01a, LM99].

11.1 Preliminaries and Definitions

Denote by Q a geodesic plane, an equidistant surface or a horosphere in $\mathbb{H}^3$. Fix an orientation ξ on Q. We define a geodesic graph on a domain of Q that extends the given one in Definition 10.4.1.

Definition 11.1.1 Let $\Omega \subset Q$ be a bounded domain and $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$. The geodesic graph of f is the set $\{\gamma_q(f(q)) : q \in \Omega\}$, where $\gamma_q = \gamma_q(s)$ is the unique geodesic orthogonal to Ω at q in the direction of $\xi(q)$ and s is the arc parameter.

Note that each point p of the graph is determined by a unique point $q \in \Omega$ and the number $f(q)$. This is because for an equidistant surface and for a horosphere, two geodesics through two distinct points of Ω and perpendicular to Ω do not intersect. If $f = 0$ along $\partial\Omega$, the boundary of the graph is $\partial\Omega$.

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_11,
© Springer-Verlag Berlin Heidelberg 2013

When Q is a horosphere, this problem has received special attention because in the upper half-space model of $\mathbb{H}^3$, a horosphere is given by a horizontal plane $L(c)$ and a geodesic graph on a domain $\Omega \subset L(c)$ is a Euclidean graph of a function $z = u(x, y)$. The function u satisfies an elliptic equation of divergence type given in Eq. (11.8) below. Moreover, we may assume that Ω is included in the plane $z = 0$ and $u = 0$ on $\partial\Omega$. In such case, $\Omega \subset \partial_\infty \mathbb{H}^3$ and the graph represents a surface in $\mathbb{H}^3$ with non-empty asymptotic boundary. We refer to [DuHi94, DuHi95, DS94, EN96, ET01, GS00, Lie00, NeSp96, Sal89, Ton96].

In this chapter we work with the Minkowski model of $\mathbb{H}^3$ (see Sect. 10.6). Umbilical surfaces in this model are determined by the intersections between $\mathbb{H}^3$ as a hyperquadric of $\mathbb{L}^4$ with hyperplanes of $\mathbb{R}^4$. Depending on the causal character of the hyperplane, we have the following description of umbilical surfaces. Let $a \in \mathbb{L}^4$ be a vector such that $\langle a, a\rangle = \varepsilon$, with $\varepsilon \in \{-1, 0, 1\}$ and let $\tau \in \mathbb{R}$. Up to isometries in $\mathbb{L}^4$, the umbilical surfaces of $\mathbb{H}^3$ are the sets

$$U_{a,\tau} = \left\{q \in \mathbb{H}^3 : \langle q, a\rangle = \tau\right\},$$

and their classification is as follows:

1. Geodesic plane: $\varepsilon = 1$ and $\tau = 0$.
2. Equidistant surface: $\varepsilon = 1$, $\tau \neq 0$.
3. Horosphere: $\varepsilon = 0$, $\tau \neq 0$.
4. Hyperbolic sphere: $\varepsilon = -1$, $|\tau| > 1$.

A unit normal field ξ on $U_{a,\tau}$ is given by

$$\xi(q) = -\lambda(a + \tau q), \quad \lambda = \frac{1}{\sqrt{\tau^2 + \varepsilon}}. \tag{11.1}$$

The Weingarten map A on $U_{a,\tau}$ with respect to ξ is $Av = (\lambda\tau)v$, $v \in T_q U_{a,\tau}$. Thus, as we know, the mean curvature is constant on $U_{a,\tau}$ and is given by

$$h = \lambda\tau = \frac{\tau}{\sqrt{\tau^2 + \varepsilon}}.$$

Then $h = 0$ for a geodesic plane, $0 < |h| < 1$ for an equidistant surface, $|h| = 1$ for a horosphere and $|h| > 1$ for a hyperbolic sphere.

Throughout this chapter, $U_{a,\tau}$ (resp. Q) stands for a geodesic plane, an equidistant surface or a horosphere in the Minkowski (resp. upper half-space) model of $\mathbb{H}^3$. We choose $\tau \geq 0$, replacing a by $-a$ if necessary. The orientation ξ on $U_{a,\tau}$ will be as in (11.1) and as a consequence, the mean curvature is $h \geq 0$.

We express the notion of a graph in the Minkowski model of $\mathbb{H}^3$. The geodesic of $\mathbb{H}^3$ through a point x and with initial velocity v is $\gamma(s) = \cosh(s)x + \sinh(s)v$. Let $\Omega \subset U_{a,\tau}$ and $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Then the graph M of f is

$$\begin{aligned}\operatorname{graph}(f) &= \left\{\cosh f(q)q + \sinh f(q)\xi(q) : q \in \Omega\right\} \\ &= \left\{\big(\cosh f(q) - h\sinh f(q)\big)q - \lambda \sinh f(q)a : q \in \Omega\right\}. \end{aligned} \tag{11.2}$$

In the scheme of the method of continuity, we have to establish a priori estimates of $|f|$ and $|\nabla f|$. We express both functions in terms of the functions $\langle p, a\rangle$ and $\langle N, a\rangle$, as was done in Euclidean space. Assume that $\partial M \subset U_{a,\tau}$, that is, $f = 0$ along $\partial\Omega$ and $\langle p, a\rangle = \tau$ for $p \in \partial M$. Using the notation $p = (q, f(q)) \in M$, with $q \in \overline{\Omega}$, we find that

$$\begin{aligned}\langle p, a\rangle &= \big(\cosh f(q) - h \sinh f(q)\big)\tau - \lambda \sinh f(q)\varepsilon \\ &= \tau \cosh f(q) - (\lambda + h\tau)\sinh f(q) = \tau \cosh f(q) - \frac{1}{\lambda}\sinh f(q).\end{aligned}$$

We relate the gradient ∇f with the function $\langle N, a\rangle$, where N is a unit normal vector field. Consider

$$\begin{aligned}N(p) = -\frac{1}{m}\Big(&\big(\cosh(f) - h\sinh(f)\big)\big(h\cosh(f) - \sinh(f)\big)q \\ &+ \lambda\cosh(f)\big(\cosh(f) - h\sinh(f)\big)a + \nabla f\Big),\end{aligned}$$

where ∇f is the gradient of f as function defined on $\overline{\Omega} \subset U_{a,\tau}$ and

$$m = \sqrt{\big(\cosh(f) - h\sinh(f)\big)^2 + |\nabla f|^2}.$$

We have the following.

1. If $U_{a,\tau}$ is a geodesic plane or an equidistant surface,

$$f(q) = \log\frac{\lambda}{(1-h)}\frac{1}{\langle p, a\rangle + \sqrt{1 + \langle p, a\rangle^2}}, \tag{11.3}$$

$$\langle N, a\rangle = -\frac{(\cosh(f) - h\sinh(f))^2}{\lambda\sqrt{(\cosh(f) - h\sinh(f))^2 + |\nabla f|^2}}. \tag{11.4}$$

2. If $U_{a,\tau}$ is a horosphere,

$$f(q) = \log\frac{\tau}{\langle p, a\rangle}, \tag{11.5}$$

$$\langle N, a\rangle = -\frac{\tau e^{-f}}{\sqrt{1 + e^{2f}|\nabla f|^2}}. \tag{11.6}$$

Let us observe that with this choice of the Gauss map N, we have $\langle N, a\rangle \le 0$ on M.

Let $\Omega \subset Q$ be a bounded domain. From Theorem 10.3.6, we know that an H-graph on Ω with boundary $\partial\Omega$ and $|H| \ne h$ is included in one of the two connected components of $\mathbb{H}^3 \setminus Q$. Furthermore, if $|H| \le h$, the graph is included in the convex side of Q. Thus, and according to the choice of τ, ξ and N, we obtain:

Corollary 11.1.2 *Let $\Omega \subset U_{a,\tau}$ be a bounded domain and let $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $f = 0$ along $\partial\Omega$. Assume that $M = \mathrm{graph}(f)$ has constant mean curvature H.*

1. *If $H = h$, then $M = \overline{\Omega}$, i.e., $f = 0$ on $\overline{\Omega}$.*
2. *If $H < h$, then $\langle p, a\rangle < \tau$ for $p \in int(M)$, i.e., $f > 0$ on Ω.*
3. *If $H > h$, then $\langle p, a\rangle > \tau$ for $p \in int(M)$, i.e., $f < 0$ on Ω.*

Moreover, if $H \geq 0$, then $\langle p, a\rangle \geq 0$, for $p \in M$.

Proof The result is a consequence of Corollary 10.1.8. It remains to consider the case $H \geq 0$. Assume, on the contrary, that $\langle x, a\rangle$ takes negative values on M. Let $p_0 \in M$ be a relative minimum of this function where $\langle p_0, a\rangle < 0$. But (2.27) gives

$$\Delta\langle x, a\rangle(p_0) = 2\langle p_0, a\rangle - 2\big\langle N(p_0), a\big\rangle \geq 2\langle p_0, a\rangle > 0,$$

a contradiction. □

When $|H| \leq 1$, we know by Proposition 10.1.4 that the size of an H-surface M spanning a given boundary curve Γ depends only on Γ. This result is obtained in the Minkowski model as follows:

Theorem 11.1.3 *Let $\Gamma \subset \mathbb{H}^3$ be a simple closed curve and $a \in \mathbb{L}^4$. Then there exists a constant $C_1 = C_1(\Gamma, a)$ depending only on Γ and a, such that if M is a compact surface immersed in $\mathbb{H}^3$ spanning Γ with constant mean curvature H, $|H| \leq 1$, then*

$$\max_{p\in M} \big|\langle p, a\rangle\big| \leq C_1.$$

Proof Let $\{e_1, \ldots, e_4\}$ be the canonical basis of $\mathbb{L}^4$. By Proposition 10.1.4, we know that there exists a hyperbolic ball whose radius depends only on Γ containing M. By a change of the center, we suppose that M lies in the ball $\{q \in \mathbb{H}^3 : \langle q, e_4\rangle \geq \tau\}$ for some $\tau < -1$ and $\tau = \tau(\Gamma)$. Then $\langle p, e_4\rangle \geq \tau$ for all $p \in M$. Since $\langle p, p\rangle = -1$ for $p \in \mathbb{H}^3$, we have for each $1 \leq i \leq 3$,

$$-\langle p, e_4\rangle^2 + \langle p, e_i\rangle^2 \leq \langle p, p\rangle = -1.$$

Then

$$\big|\langle p, e_i\rangle\big| \leq \sqrt{\langle p, e_4\rangle^2 - 1} \leq \sqrt{\tau^2 - 1} = B_1.$$

Finally, and if we write $a = a_1e_1 + \ldots + a_4e_4$, we conclude

$$\big|\langle p, a\rangle\big| \leq |a_4||\tau| + B_1\big(|a_1| + |a_2| + |a_3|\big) = C_1.$$ □

We may also argue as follows. Since $\langle HN(p) + p, HN(p) + p\rangle = H^2 - 1 \leq 0$ for each $p \in M$, the vector $HN(p) + p$ is a timelike or lightlike vector in Lorentz-Minkowski space $\mathbb{L}^4$. As $\langle e_4, e_4\rangle = -1$, we have $\langle HN(p) + p, e_4\rangle < 0$ on M or $\langle HN(p) + p, e_4\rangle > 0$ on M. We claim that the second case is not possible. If it is, suppose that $\langle N, e_4\rangle > 0$ (and so, $H > 0$). We claim the comparison principle with spheres $\langle q, e_4\rangle = \tau$, $\tau < -1$. By (11.1), the Gauss map is $\xi = \lambda(e_4 + \tau q)$ for the positive mean curvature $h = -\lambda\tau$. Then $\langle \xi, e_4\rangle = \lambda(\tau^2 - 1) = \sqrt{\tau^2 - 1} > 0$. The comparison principle would yield $H \geq h$, a contradiction. Thus $\langle HN(p) + p, e_4\rangle < 0$.

From (2.27) (taking $c=-1$) and since $\langle p, e_4\rangle < 0$,

$$\Delta\langle p, e_4\rangle = 2\big(H\big\langle N(p), e_4\big\rangle + \langle p, e_4\rangle\big) \leq 0.$$

The maximum principle implies that

$$0 > \langle p, e_4\rangle \geq \min_{p\in\Gamma}\langle p, e_4\rangle = b_1 \tag{11.7}$$

and b_1 depends only on Γ. Now the argument follows the same steps as in the proof of theorem.

11.2 Graphs on a Horosphere

Let $\Omega \subset L(c)$ be a bounded domain and let $u \in C^2(\Omega)\cap C^0(\overline{\Omega})$ be a positive function. Then the geodesic graph of u is a Euclidean graph on the horizontal plane $z=0$ given by $M=\{(q,u(q)) : q\in\overline{\Omega}\}$. Assume that $\partial M = \partial\Omega$. If Ω' is the orthogonal projection of Ω into the plane $\mathbb{R}^2=\mathbb{R}^2\times\{0\}$, the relation (10.4) says that the mean curvature H satisfies

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \frac{2}{u}\left(H - \frac{1}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{on } \Omega' \tag{11.8}$$

$$u = c \quad \text{on } \partial\Omega'. \tag{11.9}$$

This section is devoted to the proof of the following theorem.

Theorem 11.2.1 *Let $\Omega \subset U_{a,\tau}$ be a bounded domain such that $\Gamma=\partial\Omega$ is strictly mean convex. Denote by $H_\Gamma > 0$ the mean curvature of $\partial\Omega \hookrightarrow U_{a,\tau}$ computed with the inward orientation. Let H be a real number such that*

$$-H_\Gamma < H \leq 1.$$

Then there exists an H-graph on Ω with boundary $\partial\Omega$.

Observe that if $\partial\Omega$ is a circle of radius $\rho>0$ in the horosphere $L(c)$, then for each $H\in(-H_\Gamma, 1]$ there is an umbilical H-graph on Ω spanning $\partial\Omega$: see Fig. 11.1.

We begin with a result that uses the tangency principle to compare two graphs depending on the value of the mean curvature.

Proposition 11.2.2 (Monotonicity) *Let Ω_1, Ω_2 be two bounded domains of a horosphere $L(c)$ such that $\overline{\Omega_1}\subset\Omega_2$. Consider two functions $u_i\in C^2(\Omega_i)\cap C^0(\overline{\Omega_i})$ whose graphs M_i have constant mean curvatures H_i with respect to the upwards orientation, $i=1,2$. If $H_2\leq H_1\leq 1$, then $u_1\leq u_2$ on Ω_1. In particular, if $\Omega_1=\Omega_2$, then $u_1\leq u_2$.*

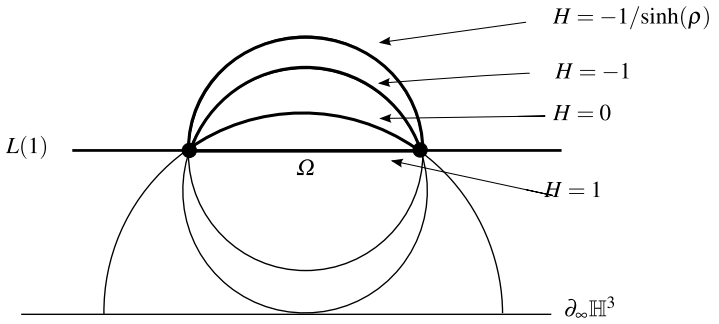

Fig. 11.1 Here $\Omega \subset L(1)$ is a round disk of radius ρ. Since $L(1)$ is isometric to $\mathbb{R}^2$, $H_\Gamma = 1/r = 1/\sinh(\rho)$, where r is the Euclidean radius. The H-graphs begin with the domain Ω itself ($H = 1$), then a domain of a geodesic plane ($H = 0$), we continue with another piece of a horosphere ($H = -1$) and we finish with the umbilical surface that it is a Euclidean hemisphere orthogonal to $L(1)$, where the value of H is $-1/\sinh(\rho)$. Recall that the orientation N in these surfaces points upwards

Proof Since $H_i \leq 1$, the surfaces M_i lie in the mean convex side of $L(c)$, that is, $M_i \subset \{(x, y, z) :\in \mathbb{R}^3_+ : z \geq c\}$. There is no loss of generality in assuming that $(0, 0, c) \in \Omega_1$. Consider the hyperbolic translations given by $\phi_t(p) = tp$, $t > 0$. Let $p_0 = (0, 0, c)$ and let $K(\Omega, p_0)$ be the Killing cylinder with respect to γ_{p_0} where γ_{p_0} is the vertical line through p_0. Since M_i is a graph on Ω_i and M_i lies over the plane $z = c$, we have $\phi_t(M_i) \subset K(\Omega_i, p_0)$ for $t > 1$. Moreover, and because $\overline{\Omega_1} \subset \Omega_2$, we see that $K(\Omega_1, p_0) \subset K(\Omega_2, p_0)$, where the inclusion is strict.

Let us take a large number t such that $\phi_t(M_2)$ does not intersect M_1. Now we bring $\phi_t(M_2)$ back to its original position by decreasing t until the first time $t_0 \geq 1$ such that $\phi_{t_0}(M_2) \cap M_1 \neq \emptyset$. Since $\overline{\Omega_1} \subset \Omega_2$, $\phi_t(\partial\Omega_2)$ does not touch M_1 for all $t > 1$. On the other hand, the tangency principle forbids an interior contact point in $\phi_{t_0}(M_2) \cap M_1$ because $H_1 \geq H_2$. Then $t_0 \leq 1$ and M_2 is above M_1 on Ω_1. □

Denote by $P(H)$ the Dirichlet problem (11.8)–(11.9) indicating the dependence of H. We apply the continuity method as in the Euclidean setting and define the set

$$\mathscr{S} = \big\{t \in [0, 1]\text{: there exists a solution } u_t \text{ of } P(1 - t + tH)\big\}.$$

First, $\mathscr{S} \neq \emptyset$ because the domain Ω is itself a graph with constant mean curvature 1 and thus $0 \in \mathscr{S}$. Now we study the C^1-estimates of the solutions of the Dirichlet problem.

A consequence of Corollary 11.1.2 and Theorem 11.1.3 is:

Corollary 11.2.3 *Let $H \in [0, 1]$ and let $\Omega \subset U_{a,\tau}$ be a bounded domain. Then there exists a constant $C_2 = C_2(\Omega)$ depending only on Ω, such that if M is an H-graph of $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $f = 0$ on Ω, then*

$$0 \leq f \leq C_2.$$

We analyze the case $H \notin [0,1]$.

Theorem 11.2.4 *Let $\Omega \subset U_{a,\tau}$ be a bounded domain and $H \notin [0,1]$. Then there exists a constant $C_3 = C_3(H)$ depending only on H such that if $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $f = 0$ on $\partial\Omega$ and $M = \operatorname{graph}(f)$ is an H-graph, we have $|f| \le C_3$. Moreover,*

1. $0 \le f \le C_3$, *if* $H < 0$.
2. $-C_3 \le f \le 0$, *if* $H > 1$.

Proof Combining (2.27) and (2.28) (letting $c = -1$), we deduce

$$\Delta\big(H\langle p, a\rangle + \langle N, a\rangle\big) = -\big(|\sigma|^2 - 2H^2\big)\langle N, a\rangle \ge 0. \tag{11.10}$$

The maximum principle for the function $H\langle p, a\rangle + \langle N, a\rangle$ says

$$H\langle p, a\rangle + \big\langle N(p), a\big\rangle \le \max_{\partial M}\big(H\tau + \langle N, a\rangle\big) \le H\tau. \tag{11.11}$$

Now we have

$$\begin{aligned} 0 = |a|^2 &= \big|a^\top\big|^2 + \langle N, a\rangle^2 - \langle p, a\rangle^2 \ge \langle N, a\rangle^2 - \langle p, a\rangle^2 \\ &= \big(\langle N, a\rangle + \langle p, a\rangle\big)\big(\langle N, a\rangle - \langle p, a\rangle\big), \end{aligned}$$

where $a^\top$ is the tangent part of a on M. Since $\langle N(p), a\rangle - \langle p, a\rangle \le 0$, we have

$$\big\langle N(p), a\big\rangle + \langle p, a\rangle \ge 0. \tag{11.12}$$

From (11.11) and (11.12), we get

$$(H-1)\langle p, a\rangle \le H\tau. \tag{11.13}$$

By (11.5), the inequality (11.13) yields the desired C^0-estimates of f by letting

$$C_3 = \log\frac{H-1}{H}.$$

Moreover, it is straightforward from (11.13) that $f \ge 0$ if $H < 0$ and $f \le 0$ if $H > 1$. □

The height estimates of this result can be seen as an extension in our context of Serrin's estimate given in Theorem 2.3.5. We have proved in Theorem 11.2.4 that an H-graph of a function f defined on a domain Ω of a horosphere $U_{a,\tau}$, with $f = 0$ on $\partial\Omega$ satisfies

$$0 \le |f| \le \log\left|\frac{H-1}{H}\right|,$$

whenever $H < 0$ or $H > 1$.

For the estimate of $|\nabla f|$, we have:

Theorem 11.2.5 *Let $\Omega \subset U_{a,\tau}$ be a bounded domain such that $\Gamma = \partial\Omega$ is strictly mean convex. Consider an H-graph M of $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $f = 0$ on $\partial\Omega$. If*

$$-H_\Gamma < H \leq 1,$$

then there exists a constant $C_4 = C_4(\Omega, H)$, depending only on Ω and H, such that

$$\sup_{\Omega} |\nabla f| \leq C_4.$$

Proof (Compare with Theorem 8.3.1) The estimate of $|\nabla f|$ is given provided we have a priori bounds of the function $\langle N, a\rangle$ according to (11.6). From Eq. (11.10), there exists a $q \in \partial M$ such that

$$H\langle p, a\rangle + \langle N, a\rangle \leq H\tau + \big\langle N(q), a\big\rangle. \tag{11.14}$$

Furthermore, by the maximum principle, at the point q we find that

$$H\langle \nu_q, a\rangle + \big\langle (dN)_q \nu_q, a\big\rangle \leq 0,$$

where ν is the inward unit conormal vector of M along ∂M ($\nu_q = \nu(q)$). Using $H \leq 1$, we know from Corollary 11.1.2 that $\langle p, a\rangle \leq \tau$ and then $\langle \nu, a\rangle \leq 0$ on $\partial\Omega$. This implies

$$\big(H - \sigma(\nu_q, \nu_q)\big)\langle \nu_q, a\rangle \leq 0.$$

From this inequality and since $\langle \nu_q, a\rangle \leq 0$, we obtain

$$\sigma(\nu_q, \nu_q) \geq H. \tag{11.15}$$

At the boundary point q we see that

$$\sigma(\nu_q, \nu_q) = \sigma^\Gamma(\nu_q, \nu_q)\big\langle N(q), \eta(q)\big\rangle - \frac{\langle N, a\rangle}{\tau}, \tag{11.16}$$

where σ^Γ denotes the second fundamental form of Γ as a curve of $U_{a,\tau}$ and η is the inward unit normal field to Γ in $U_{a,\tau}$. Since $\xi = -p - a/\tau$ is the unit normal field on $U_{a,\tau}$,

$$1 = \langle \xi, \xi\rangle = \langle \nu, \xi\rangle^2 + \langle N, \xi\rangle^2 = \frac{\langle \nu, a\rangle^2}{\tau^2} + \langle N, \xi\rangle^2.$$

This formula, together with $1 = \langle N, N\rangle = \langle N, \eta\rangle^2 + \langle N, \xi\rangle^2$, assures us that

$$\langle N, \eta\rangle = \frac{\langle \nu, a\rangle}{\tau}.$$

Combining (11.15) and (11.16) and since $H_\Gamma > 0$, we have

$$\big\langle N(q), a\big\rangle \leq H_\Gamma(q)\langle \nu_q, a\rangle - \tau H. \tag{11.17}$$

1. *Case* $0 < H < 1$. The inequalities (11.14) and (11.17) yield

$$H\langle p, a\rangle + \big\langle N(p), a\big\rangle \leq 0 \tag{11.18}$$

on M. As a consequence of Corollary 11.2.3, we deduce

$$\big\langle N(p), a\big\rangle \leq -H\langle p, a\rangle \leq -H\tau e^{-C_2} = B_2 < 0$$

and B_2 depends only on Ω and H.

2. *Case* $-H_\Gamma < H \leq 0$. As $\langle a, a\rangle = 0$, we have

$$\langle v, a\rangle = -\sqrt{\tau^2 - \langle N, a\rangle^2}.$$

Then (11.17) gives

$$\big(1 + H_\Gamma(q)^2\big)\big\langle N(q), a\big\rangle^2 + 2\tau H\big\langle N(q), a\big\rangle + \tau^2\big(H^2 - H_\Gamma(q)^2\big) \geq 0. \tag{11.19}$$

We view (11.19) as a quadratic inequality on $\langle N(q), a\rangle$. Using $H_\Gamma^2 > H^2$, the left hand side has one positive and one negative root. As $\langle N, a\rangle < 0$, we deduce

$$\big\langle N(q), a\big\rangle \leq \frac{\tau}{1 + H_\Gamma(q)^2}\big(H - H_\Gamma(q)\sqrt{1 + H_\Gamma(q)^2 - H^2}\big). \tag{11.20}$$

The assumption on H assures us that the right-hand side of (11.20) is negative. Consider

$$\kappa_0 = \min_{p\in\partial\Omega} H_\Gamma(p), \qquad \kappa_1 = \max_{p\in\partial\Omega} H_\Gamma(p).$$

Now (11.14), (11.20) and $-H_\Gamma < H \leq 0$ imply

$$\begin{aligned}\langle N, a\rangle &\leq H\big(\tau - \langle p, a\rangle\big) + \frac{\tau}{1 + H_\Gamma(q)^2}\big(H - H_\Gamma\sqrt{1 + H_\Gamma(q)^2 - H^2}\big)\\ &\leq \frac{\tau}{1 + H_\Gamma(q)^2}\big(H - H_\Gamma(q)\sqrt{1 + H_\Gamma(q)^2 - H^2}\big)\\ &\leq \frac{\tau}{1 + \kappa_1^2}\big(H - \kappa_0\sqrt{1 + \kappa_0^2 - H^2}\big) = B_3 < 0,\end{aligned}$$

where B_3 depends only on Ω and H.

Summarizing, we have on Ω,

$$\begin{aligned}&\langle N, a\rangle \leq B_2 < 0, \quad \text{if } 0 < H < 1.\\ &\langle N, a\rangle \leq B_3 < 0, \quad \text{if } -H_\Gamma < H \leq 0.\end{aligned}$$

From these estimates, Corollary 11.2.3, Theorem 11.2.4 and the expression of $\langle N, a\rangle$ in (11.4), we conclude that there exists a constant C_4, depending only on Ω and H, such that $|\nabla f| \leq C_4$. □

Proof of Theorem 11.2.5 The above estimates of $|f|$ and $|\nabla f|$ imply that the set $\mathscr{S}$ is closed in $[0,1]$. In order to prove that $\mathscr{S}$ is open, it suffices to show that the kernel of the linearized operator of the mean curvature is trivial (see Lemma 8.1.4 and Remark 8.1.5). In this case, the Jacobioperator is $L = \Delta + |\sigma|^2 - 2$. The function $\langle N, a\rangle$ is negative and by (2.28),

$$L\big(\langle N, a\rangle\big) = -2\big(H\langle p, a\rangle + \langle N, a\rangle\big).$$

If $0 < H < 1$, we see that $L(\langle N, a\rangle) \geq 0$ by (11.18). When $H \leq 0$, Theorem 11.2.4 gives $f \geq 0$ and (11.5) implies $L(\langle N, a\rangle) > 0$ because $\langle p, a\rangle > 0$ and $\langle N, a\rangle < 0$. In both cases, we have $L(\langle N, a\rangle) \geq 0$, which proves that $\mathscr{S}$ is an open set in $[0,1]$. □

11.3 Graphs on a Geodesic Plane and an Equidistant Surface

In this section we study the Dirichlet problem for geodesic graphs on a domain of a geodesic plane or an equidistant surface $U_{a,\tau}$. For an appropriate orientation, suppose the mean curvature of $U_{a,\tau}$ is $h \geq 0$. The main result is:

Theorem 11.3.1 *Let $\Omega \subset U_{a,\tau}$ be a bounded domain such that $\Gamma = \partial\Omega$ is strictly mean convex. Suppose H a real number such that*

$$-H_\Gamma < H \leq h,$$

where H_Γ denotes the mean curvature of $\partial\Omega \hookrightarrow U_{a,\tau}$ with respect to the inward orientation. Then there exists a geodesic H-graph on Ω with boundary $\partial\Omega$. Furthermore, if $H_\Gamma > 1$, the interval of existence extends to $-H_\Gamma < H \leq 1$.

Starting from the trivial solution $f = 0$ on Ω (for $H = h$), which corresponds to the domain Ω itself, appropriate C^1-estimates will allow us to blow up $f = 0$ obtaining solutions for each H' until we arrive at the value $H' = H$. As in the preceding section, and using (11.3) and (11.4), we derive C^1-estimates for f provided that there are bounds for the functions $\langle p, a\rangle$ and $\langle N, a\rangle$.

We first study height estimates of f. We start by considering the case $|H| \leq 1$. A direct consequence of Corollary 11.1.2 and Theorem 11.1.3 is:

Theorem 11.3.2 *Let $H \in \mathbb{R}$ satisfy $|H| \leq 1$. Let $\Omega \subset U_{a,\tau}$ be a bounded domain. Then there exists a constant $C_1 = C_1(\Omega) > 0$ such that if $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $f = 0$ on $\partial\Omega$ and $M = \operatorname{graph}(f)$ is an H-graph, then $|f| \leq C_1$. Furthermore, $0 \leq f \leq C_1$ (resp. $-C_1 \leq f \leq 0$) if $-1 \leq H \leq h$ (resp. $h \leq H \leq 1$).*

We also need a priori estimates of $|f|$ in the range $|H| > 1$.

Theorem 11.3.3 *Let $H \in \mathbb{R}$ satisfy $|H| > 1$. Let $\Omega \subset U_{a,\tau}$ be a bounded domain. There exist constants $C_2 = C_2(\Omega, H)$ and $C_3 = C_3(\Omega, H)$ such that if*

$f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $f = 0$ on $\partial\Omega$ and $M = \text{graph}(f)$ is an H-graph, then

$$C_2 \leq f \leq C_3.$$

Proof By (11.10), the maximum principle gives again

$$H\langle p, a\rangle + \langle N, a\rangle \leq \max_{q\in\partial M}\big(H\langle p, a\rangle + \langle N, a\rangle\big) \leq H\tau. \tag{11.21}$$

Since $\langle a, a\rangle = 1$, for each $p \in M$ we have

$$\langle a, a\rangle = 1 = \big\langle a^T, a^T\big\rangle^2 + \langle N, a\rangle^2 - \langle p, a\rangle^2 \geq \langle N, a\rangle^2 - \langle p, a\rangle^2,$$

where a^T denotes again the tangent part of the vector a on M. Then

$$\langle N, a\rangle^2 \leq 1 + \langle p, a\rangle^2.$$

Inequality (11.21) yields

$$H\big(\langle p, a\rangle - \tau\big) \leq -\langle N, a\rangle \leq \sqrt{1 + \langle p, a\rangle^2}. \tag{11.22}$$

From Corollary 11.1.2, if $H > 1$ then $\langle p, a\rangle \geq \tau$ and if $H < -1$, we see that $\langle p, a\rangle \leq \tau$. Squaring (11.22) we get

$$H^2\big(\langle p, a\rangle - \tau\big)^2 \leq 1 + \langle p, a\rangle^2,$$

or equivalently,

$$\big(H^2 - 1\big)\langle p, a\rangle^2 - 2\tau H^2\langle p, a\rangle + H^2\tau^2 - 1 \leq 0.$$

Consequently

$$\frac{\tau H^2 - \sqrt{H^2\tau^2 + H^2 - 1}}{H^2 - 1} \leq \langle p, a\rangle \leq \frac{\tau H^2 + \sqrt{H^2\tau^2 + H^2 - 1}}{H^2 - 1}. \tag{11.23}$$

Hence we have lower and upper bounds for the function $\langle p, a\rangle$ and therefore, the corresponding C^0-estimates of the function f that defines the graph. This proves the theorem. □

Inequalities (11.23) may be viewed again as the generalization to the hyperbolic setting, with $|H| > 1$, of Serrin's height estimate. If the boundary is included in a geodesic plane ($\tau = 0$), we have

$$-\frac{1}{\sqrt{H^2 - 1}} \leq \langle p, a\rangle \leq \frac{1}{\sqrt{H^2 - 1}}.$$

By (11.3), the function f satisfies

$$\big|f(q)\big| \leq \text{arc sinh}\left(\frac{1}{\sqrt{H^2 - 1}}\right).$$

This estimate appeared in [KKMS92]. In addition, we derive from (11.23) a similar estimate of the height of a geodesic graph on a domain of an equidistant surface, where now $\langle p, a\rangle = -\sinh(f + \log(\tau(1-h)/h))$.

We obtain the gradient estimates for the solution of the Dirichlet problem.

Theorem 11.3.4 *Consider a bounded domain $\Omega \subset U_{a,\tau}$ such that $\Gamma = \partial\Omega$ is strictly mean convex. Let $H \in \mathbb{R}$ satisfy $-H_\Gamma < H < H_\Gamma$. Then there exists a constant $C_4 = C_4(\Omega, H)$ such that if $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $f = 0$ on $\partial\Omega$ and $M = \operatorname{graph}(f)$ is an H-graph, then*

$$\sup_{\Omega} |\nabla f| \le C_4.$$

Proof According to the relation between $|\nabla f|$ and the function $\langle N, a\rangle$ given in (11.4), it suffices to estimate $\langle N, a\rangle$. We distinguish two cases.

Case 1. $-H_\Gamma < H \le h$.

From Corollary 11.1.2 we see that $\langle p, a\rangle \le \tau$. Hence it follows that $\langle \nu, a\rangle \le 0$ in $\partial\Omega$, where ν denotes the inner conormal vector of M along $\partial\Omega$. In view of (11.21), there exists a $q \in \partial\Omega$ such that the function $H\langle p, a\rangle + \langle N, a\rangle$ assumes its maximum, that is,

$$H\langle p, a\rangle + \langle N, a\rangle \le H\tau + \big\langle N(q), a\big\rangle. \tag{11.24}$$

Furthermore (11.21) implies

$$H\langle v_q, a\rangle + \langle dN_q v_q, a\rangle \le 0,$$

where N is a unit normal field on M such that $\langle N, a\rangle \le 0$. Hence

$$\big(H - \sigma(v_q, v_q)\big)\langle v_q, a\rangle \le 0,$$

where σ is the second fundamental form of M. Since $\langle \nu, a\rangle \le 0$ along $\partial\Omega$, we deduce

$$H - \sigma(v_q, v_q) \ge 0. \tag{11.25}$$

On the other hand, let α be an arc length parametrization of Γ with $\alpha(0) = q$ and η be the unit normal field to $\partial\Omega$ in $U_{a,\tau}$ pointing to Ω. Then

$$\begin{aligned}\sigma\big(\alpha'(0), \alpha'(0)\big) &= \big\langle N(q), \alpha''(0)\big\rangle \\ &= -\lambda h\big\langle N(q), a\big\rangle + \sigma^\Gamma\big(\alpha'(0), \alpha'(0)\big)\big\langle N(q), \eta(q)\big\rangle. \end{aligned}\tag{11.26}$$

Since along $\partial\Omega$

$$1 = \langle N, N\rangle = \langle N, \eta\rangle^2 + \langle N, \xi\rangle^2 = \langle N, \eta\rangle^2 + \lambda^2\langle N, a\rangle^2,$$

it follows that

$$\langle N, \eta\rangle = -\sqrt{1 - \lambda^2\langle N, a\rangle^2}. \tag{11.27}$$

Combining (11.26) and (11.27), we get

$$\sigma\big(\alpha'(0),\alpha'(0)\big) = -\lambda h\big\langle N(q),a\big\rangle - H_\Gamma(q)\sqrt{1-\lambda^2\big\langle N(q),a\big\rangle^2}.$$

Then (11.25) yields $\sigma(\alpha'(0),\alpha'(0)) - H \geq 0$, that is,

$$-\lambda h\big\langle N(q),a\big\rangle - H_\Gamma(q)\sqrt{1-\lambda^2\big\langle N(q),a\big\rangle^2} \geq H \tag{11.28}$$

or equivalently

$$H_\Gamma(q)\sqrt{1-\lambda^2\big\langle N(q),a\big\rangle^2} \leq -\lambda h\big\langle N(q),a\big\rangle - H.$$

Squaring, we deduce

$$\lambda^2\big(h^2+H_\Gamma(q)^2\big)\big\langle N(q),a\big\rangle^2 + 2\lambda h H\big\langle N(q),a\big\rangle + H^2 - H_\Gamma(q)^2 \geq 0. \tag{11.29}$$

We view this expression as a quadratic inequality on $\langle N(q),a\rangle$. Let $\rho_1 \leq \rho_2$ be the two roots of the left-hand side of in (11.29):

$$\rho_1,\rho_2 = \frac{-hH \pm H_\Gamma(q)\sqrt{h^2+H_\Gamma(q)^2-H^2}}{\lambda(h^2+H_\Gamma(q)^2)}. \tag{11.30}$$

Inequality (11.29) implies that $\langle N(q),a\rangle \leq \rho_1$ or $\langle N(q),a\rangle \geq \rho_2$. Let

$$\kappa_0 = \min_{p\in\partial\Omega} H_\Gamma(p), \qquad \kappa_1 = \max_{p\in\partial\Omega} H_\Gamma(p).$$

We distinguish three cases according the value of H:

(a) *Case* $H \leq 0$. We point out that in this case $U_{a,\tau}$ is a geodesic plane, since $h=0$. As $H \leq 0$, $\rho_2 > 0$ and since $\langle N(q),a\rangle < 0$, $\langle N(q),a\rangle \leq \rho_1$. The assumption $H_\Gamma(q) > -H$ implies $\rho_1 < 0$. From (11.24), we obtain

$$\begin{aligned}\langle N,a\rangle &\leq H\big(\tau - \langle p,a\rangle\big) + \big\langle N(q),a\big\rangle \leq \big\langle N(q),a\big\rangle \\ &\leq \frac{-hH - H_\Gamma(q)\sqrt{h^2+H_\Gamma(q)^2-H^2}}{\lambda(h^2+H_\Gamma(q)^2)}.\end{aligned}$$

Then we verify directly that

$$\begin{aligned}\langle N,a\rangle &\leq \frac{-hH-\kappa_0\sqrt{h^2+\kappa_0^2-H^2}}{\lambda(h^2+\kappa_1^2)} \\ &\leq \frac{\kappa_0(h-\sqrt{h^2+\kappa_0^2-H^2})}{\lambda(h^2+\kappa_1^2)}.\end{aligned} \tag{11.31}$$

(b) *Case* $0 < H \le h$. We view ρ_1 and ρ_2 as two continuous functions in the variable H. Since $\partial\Omega$ is mean convex, we have $\rho_1(0) < 0 < \rho_2(0)$. By continuity, there exists a (small) constant $h_0 = h_0(\Omega) < h$, such that $\rho_1(H) < 0 < \rho_2(H)$ for all $H \in [0, h_0]$. Hence $\langle N(q), a\rangle \le \rho_1$. For each $H \in [0, h_0]$ and by virtue of (11.24) and (11.30), we find that

$$\begin{aligned}\langle N, a\rangle &\le H\tau - H\langle p, a\rangle + \big\langle N(q), a\big\rangle \le H\tau + \big\langle N(q), a\big\rangle \\ &\le \frac{Hh(h^2 + H_\Gamma(q)^2) - hH - H_\Gamma(q)\sqrt{h^2 + H_\Gamma(q)^2 - H^2}}{\lambda(h^2 + H_\Gamma(q)^2)} \\ &\le \frac{Hh(h^2 + \kappa_1^2) - \kappa_0^2}{\lambda(h^2 + \kappa_1^2)} \end{aligned} \tag{11.32}$$

where in the second inequality we apply Corollary 11.1.2 to ensure that $\langle p, a\rangle \ge 0$. Choose h_0 sufficiently small so that

$$2h_0 h\big(h^2 + \kappa_1^2\big) - \kappa_0^2 < 0. \tag{11.33}$$

Then (11.32) becomes

$$\langle N, a\rangle \le \frac{h_0 h(h^2 + \kappa_1^2) - \kappa_0^2}{\lambda(h^2 + \kappa_1^2)} < 0. \tag{11.34}$$

(c) *Case* $H \in [h_0, h]$. By (11.28), we have $-\lambda h\langle N(q), a\rangle \ge H$ or equivalently $\langle N(q), a\rangle \le -H/(\lambda h)$. Corollary 11.1.2 and the inequality (11.24) yield

$$\begin{aligned}\langle N, a\rangle &\le H\left(\tau - \langle p, a\rangle - \frac{1}{\lambda h}\right) = H\left(\frac{h^2 - 1}{\lambda h} - \langle p, a\rangle\right) \\ &\le H\frac{h^2 - 1}{\lambda h} \le h_0\frac{h^2 - 1}{\lambda h} = -\frac{h_0}{\tau} < 0. \end{aligned} \tag{11.35}$$

Case 2. $h \le H < H_\Gamma$.

The proof is similar. Now $\langle p, a\rangle \ge \tau$. With appropriate modifications in (11.25), (11.27) and (11.28), we derive the same inequality (11.29). Since ρ_2 is positive again (because $H_\Gamma > H$), we see that $\langle N(q), a\rangle \le \rho_1$. As in (11.31), we have

$$\begin{aligned}\langle N, a\rangle &\le H\tau - H\langle p, a\rangle + \big\langle N(q), a\big\rangle \le H\tau + \big\langle N(q), a\big\rangle \\ &\le \frac{-hH - \kappa_0\sqrt{h^2 + \kappa_0^2 - H^2}}{\lambda(h^2 + \kappa_1^2)} \le -\frac{h\kappa_0}{\lambda(h^2 + \kappa_1^2)} = -\frac{\tau\kappa_0}{h^2 + \kappa_1^2}. \end{aligned} \tag{11.36}$$

Set

$$C = \max\left\{\frac{\kappa_0(h - \sqrt{h^2 + \kappa_0^2 - H^2})}{\lambda(h^2 + \kappa_1^2)}, \frac{2h_0 h(h^2 + \kappa_1^2) - \kappa_0^2}{\lambda(h^2 + \kappa_1^2)}, -\frac{h_0}{\tau}, -\frac{\tau\kappa_0}{h^2 + \kappa_1^2)}\right\}.$$

This constant is negative and depends only on Ω and H. Recall that if $U_{a,\tau}$ is a geodesic plane, we set

$$C = -\frac{\kappa_0\sqrt{\kappa_0^2 - H^2}}{\kappa_1^2}.$$

Now we obtain that $\langle N, a\rangle \leq C$. This follows from Theorems 11.3.2 and 11.3.3 the desired estimates of $|\nabla f|$. □

Proof of Theorem 11.3.1

Consider the set $\mathscr{S}$ defined by

$$\mathscr{S} = \big\{H \in (-H_\Gamma, h] : \text{ there exists an } H\text{-graph on } \Omega \text{ with boundary } \partial\Omega\big\}.$$

Since $h \in \mathscr{S}$, we have $\mathscr{S} \neq \emptyset$. We show that $\mathscr{S}$ is open as in the case of the previous section. If L is the Jacobi operator, the function $\langle N, a\rangle$ satisfies

$$L\big(\langle N, a\rangle\big) = -2\big(H\langle p, a\rangle + \langle N, a\rangle\big). \tag{11.37}$$

If $H \leq 0$, we get directly $L(\langle N, a\rangle) \geq 0$. If $H \in (0, h_0]$, Eqs. (11.33)–(11.34) imply

$$H\langle p, a\rangle + \langle N, a\rangle \leq H\tau + \frac{h_0 h(h^2 + \kappa_1^2) - \kappa_0^2}{\lambda(h^2 + \kappa_1^2)} < 0.$$

If $H \in (h_0, h]$, (11.35) gives

$$H\langle p, a\rangle + \langle N, a\rangle \leq H\left(\tau - \frac{1}{\lambda h}\right) < 0.$$

In conclusion $L(\langle N, a\rangle) > 0$. Therefore Remark 8.1.5 applies, showing that $\mathscr{S}$ is an open set in $[0, 1]$.

Finally, the proof is completed by means of the continuity method provided a priori C^1-bounds independent of H are established for each solution with $-\kappa_0 < H \leq h$. We turn our attention to the upper bound for $\langle N, a\rangle$ in Theorem 11.3.4.

1. *Case* $-1 \leq -\kappa_0$. Theorem 11.3.2 provides C^0-estimates depending only on Ω. On the other hand, if $H \leq 0$, the inequality (11.31) yields

$$\langle N, a\rangle \leq \frac{\kappa_0(h - \sqrt{h^2 + \kappa_0^2 - H^2})}{\lambda(h^2 + \kappa_1^2)}.$$

Set

$$C' = \max\left\{\frac{-\kappa_0(h - \sqrt{h^2 + \kappa_0^2 - H^2})}{\lambda(h^2 + \kappa_1^2)}, \frac{h_0}{\tau}, \frac{\kappa_0^2 - h_0 h(h^2 + \kappa_1^2)}{\lambda(h^2 + \kappa_1^2)}\right\}.$$

When $U_{a,\tau}$ is a geodesic plane, take

$$C' = \frac{\kappa_0\sqrt{\kappa_0^2 - H^2}}{\kappa_1^2}.$$

In both cases, the number $C' > 0$ depends only on Ω and H and satisfies $\langle N, a\rangle \leq -C'$ and hence we obtain the C^1-estimates of f. Since $\mathscr{S}$ is an open set, the continuity method implies that for each H, $-\kappa_0 < H \leq h$, there exists a function f on Ω vanishing on its boundary and whose graph has constant mean curvature H.

2. *Case* $-\kappa_0 < -1$. The previous construction for $-1 \leq -\kappa_0$ guarantees the existence of graphs with constant mean curvature until the value $H = -1$. Since $\mathscr{S}$ is open, in a neighborhood of $H = -1$, there exists an interval of solutions, namely $[-1-\delta, -1]$, where $\delta = \delta(\Omega) > 0$. Let $-\kappa_0 < -1 - \delta$. The bound C' for $\langle N, a\rangle$ holds as above. Then for each graph on Ω whose mean curvature H satisfies $-\kappa_0 < H \leq -1 - \delta$, Corollary 11.1.2 and estimate (11.23) imply

$$\tau \geq \langle p, a\rangle \geq \frac{\tau - \sqrt{\kappa_0^2\tau^2 + \kappa_0^2 - 1}}{H^2 - 1} \geq \frac{\tau - \sqrt{\kappa_0^2\tau^2 + \kappa_0^2 - 1}}{\delta^2 + 2\delta}.$$

Consequently, we have obtained C^0-bounds (independent of H) in the interval $(-\kappa_0, -1-\delta]$.

Finally, it remains to examine the final statement of Theorem 11.3.1. Assume $\kappa_0 > 1$. Let $\mathscr{S}$ be as in the definition but changing the interval of H to $(-H_\Gamma, 1]$. The C^0-bounds are derived in Theorem 11.3.2 and the C^1-estimates hold as above, since in the interval $[h, 1]$ we have

$$\langle N, a\rangle \leq -\frac{\tau\kappa_0}{h^2 + \kappa_1^2}.$$

In order to prove that $[h, 1]$ is open in $\mathscr{S}$, we apply Remark 8.1.5. We only have to verify that $L(\langle N, a\rangle) \geq 0$. Since $H_\Gamma > 1$, inequality (11.36) implies

$$\begin{aligned} H\langle p, a\rangle + \langle N, a\rangle &\leq H\tau - \frac{Hh + H_\Gamma(q)\sqrt{h^2 + H_\Gamma(q)^2 - H^2}}{\lambda(h^2 + H_\Gamma(q)^2)} \\ &\leq \frac{Hh(h^2 + H_\Gamma(q)^2 - 1) - H_\Gamma(q)\sqrt{h^2 + H_\Gamma(q)^2 - 1}}{\lambda(h^2 + H_\Gamma(q)^2)} \leq 0. \end{aligned}$$

By (11.37), $L(\langle N, a\rangle) \geq 0$ and this completes the proof of Theorem 11.3.1. □

Chapter 12
Constant Mean Curvature Spacelike Surfaces in Lorentz-Minkowski Space

A spacelike surface in Lorentz-Minkowski space $\mathbb{L}^3$ is a surface whose induced metric is Riemannian and thus its study is similar to that of a surface of Euclidean space. The importance of constant mean curvature spacelike surfaces or, more generally, cmc hypersurfaces in Lorentzian spaces, is well known in physics because of their role in general relativity where we are interested in finding real-valued functions on a given spacetime, all of whose level sets have constant mean curvature. Then the mean curvature function may be used as a global time coordinate and provides a time gauge, which it is important in the study of singularities, the positivity of mass and gravitational radiation. A summary of these topics can be found in the survey papers [Bar87, CY80, MT80].

In this chapter we focus on compact spacelike surfaces in $\mathbb{L}^3$ with constant mean curvature spanning a given boundary curve. Again we are interested in studying the influence of the geometry of the boundary on the shape of the surface. As in hyperbolic space, we begin this chapter by recalling basic notions of the geometry of $\mathbb{L}^3$ with a special emphasis on spacelike surfaces. Although the mean curvature equation for such a surface is again elliptic, we shall encounter some differences with the Euclidean setting. For example, a flux formula is derived and which is formally identical to that for Euclidean space, however we do not obtain similar consequences. This is due to the fact that for the formula measures the flux of the surface along a spatial direction, but now the metric in $\mathbb{L}^3$ is Lorentzian, which does not allow us to estimate the inner product of two vectors. In contrast, and in the case that the boundary is a circle, we obtain that umbilical surfaces are the only compact spacelike cmc surfaces spanning a circle. This greatly differs from what happens in Euclidean space.

We conclude the chapter with the Dirichlet problem for the mean curvature equation for spacelike graphs. Again this differs from the Euclidean case. For example, given a bounded domain $\Omega \subset \mathbb{R}^2$, the solvability of the Dirichlet problem in $\mathbb{L}^3$ is assured for arbitrary boundary data and values of H. The reason is based on the properties of the rotational cmc surfaces in $\mathbb{L}^3$, which are graphs over all $\mathbb{R}^2$ except at one point, where the surface presents a conical singularity. This provides good barriers for the necessary C^1-estimates.

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7_12,
© Springer-Verlag Berlin Heidelberg 2013

12.1 Basics on Lorentz-Minkowski Space

In this section we review several elementary facts inLorentz-Minkowski space, specially the definition of a spacelike surface and some of its properties. We refer to O'Neill [One83] and Weinstein [Wei96] for further details. Consider the three-dimensional Lorentz-Minkowski space $\mathbb{L}^3$, which is defined as the vector space $\mathbb{R}^3$ equipped with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2,$$

where (x_1, x_2, x_3) are the canonical coordinates in $\mathbb{R}^3$. Then $\langle \cdot, \cdot \rangle$ is a non-degenerate metric of index 1 classifying the vectors of $\mathbb{R}^3$ into three types.

Definition 12.1.1 A vector $v \in \mathbb{L}^3$ is said to be:

1. spacelike if $\langle v, v \rangle > 0$ or $v = 0$;
2. timelike if $\langle v, v \rangle < 0$;
3. lightlike if $\langle v, v \rangle = 0$ and $v \neq 0$.

The modulus of v is $|v| = \sqrt{|\langle v, v \rangle|}$.

The set $x_1^2 + x_2^2 - x_3^2 = 0$ of all lightlike vectors and the origin is called the *light-cone* of $\mathbb{L}^3$.

We shall also work with the usual affine structure in $\mathbb{R}^3$, the vector space that supports $\mathbb{L}^3$. Horizontal directions are indicated by the (x_1, x_2) coordinates and the vertical direction is given by the x_3-axis. Let

$$e_3 = (0, 0, 1),$$

which is a unit timelike vector and will play an important role in this chapter.

In Lorentz-Minkowski space $\mathbb{L}^3$ there does not exist a Cauchy-Schwarz inequality for the inner product of two arbitrary vectors except when they are timelike. If $u, v \in \mathbb{L}^3$ are timelike vectors, then $\langle u, v \rangle \neq 0$. Then $|\langle u, v \rangle| \geq |u||v|$ is valid and equality holds if and only if u and v are proportional. In fact, if $\langle u, v \rangle < 0$, then $\langle u, v \rangle \leq -|u||v|$.

A vector subspace $U \subset \mathbb{R}^3$ is said to be *spacelike* (resp. *timelike*, *lightlike*) if the induced metric on U is positive definite (resp. non-degenerate of index 1, degenerate and $U \neq \{0\}$). Any vector subspace belongs to one of the above three types. Moreover, U is spacelike (resp. timelike, lightlike) if its orthogonal subspace $U^\perp$ is timelike (resp. spacelike, lightlike).

Definition 12.1.2 Let M be a curve or a surface. An immersion $x : M \to \mathbb{L}^3$ is said to be spacelike if the induced metric on M is positive-definite.

We also say that M is a spacelike curve or a spacelike surface, depending on the dimension of M. As usual, we denote the induced metric on M by $\langle \cdot, \cdot \rangle$. If $x : M \to \mathbb{L}^3$ is a curve, the spacelike condition means that $\langle x'(s), x'(s) \rangle > 0$ where

$x = x(s)$ is a parametrization of the curve. In a spacelike surface, every tangent plane T_pM is spacelike which is equivalent to the property that any orthogonal vector to M is timelike. Since the induced metric on M is Riemannian, we can define on M concepts such as distance, length and area. However, the spacelike condition is strong, as shown in the following results.

Proposition 12.1.3 *There are no closed spacelike surfaces in $\mathbb{L}^3$. In particular, if $x : M \to \mathbb{L}^3$ is a spacelike immersion of a compact surface M, then $\partial M \neq \emptyset$.*

Proof Let $v \neq 0$ be a spacelike vector of $\mathbb{L}^3$. Define on M the function $f(p) = \langle x(p), v\rangle$. Since M is compact, there is a point $p \in M$ where f attains the maximum. If the surface M is closed, the point p is interior, in particular, $(df)_p(v) = 0$ for all $v \in T_pM$. Thus $T_pM = \langle v\rangle^{\perp}$. Since v is spacelike, T_pM is a timelike plane, yielding a contradiction. □

Proposition 12.1.4 *Let $\Gamma \subset \mathbb{L}^3$ be a closed spacelike curve included in a plane P. Then P is a spacelike plane.*

Proof We may suppose that P is a vector plane. If P is not a spacelike plane, there exists a lightlike vector $v \in P$. Let $\alpha : \mathbb{S}^1 \to \Gamma$ be a parametrization of Γ. Define on $\mathbb{S}^1$ the function $f(s) = \langle \alpha(s), v\rangle$ and let $s_0 \in \mathbb{S}^1$ be a point at which f attains a maximum. Then $f'(s_0) = \langle \alpha'(s_0), v\rangle = 0$, which means that $\alpha'(s_0)$ is perpendicular to v. If the plane P is lightlike or timelike, the only orthogonal vectors to v are proportional to v. This says that $\alpha'(s_0)$ is a lightlike vector, a contradiction because α is spacelike. □

The next result states that a spacelike surface is orientable.

Proposition 12.1.5 *Let M be a surface and let $x : M \to \mathbb{L}^3$ be a spacelike immersion. Then M is orientable.*

Proof We know that M is always locally orientable and we shall define a global orientation on M. Let N be a local orientation of M. For each $p \in M$, $N(p)$ is a timelike vector. Taking into account that two timelike vectors of $\mathbb{L}^3$ are not orthogonal, and since e_3 is timelike, $\langle N(p), e_3\rangle \neq 0$. Given $p_0 \in M$, let V_0 be a neighborhood of p_0 such that $\langle N, e_3\rangle < 0$ on V_0. This allows us, by connectedness, to choose a global orientation N on M such that $\langle N, e_3\rangle < 0$ on M, which proves the orientability of M. □

Since there are two possible orientations, we say that M is *future oriented* if $\langle N, e_3\rangle < 0$ on M and, in such case, we say that N pointsto the *future direction*. With this choice, the horizontal planes $x_3 = t$ are future directed when the orientation points upwards, i.e., $N = e_3$.

Given a surface M, we know that an immersion $x : M \to \mathbb{R}^3$ is locally a graph on one of the three coordinate planes. If we endow on $\mathbb{R}^3$ the Lorentzian metric,

for a spacelike surface we can be precise with respect to what coordinate plane is a graph.

Proposition 12.1.6 *A spacelike surface in $\mathbb{L}^3$ is locally a graph on the plane $x_3 = 0$.*

Proof Let $x : M \to \mathbb{L}^3$ be a spacelike immersion and $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ the projection $\pi(x_1, x_2, x_3) = (x_1, x_2)$. Consider $\tilde{x} = \pi \circ x : M \to \mathbb{R}^2$. If $v = (v_1, v_2, v_3) \in T_pM$, then

$$\left\langle (d\tilde{x})_p(v), (d\tilde{x})_p(v) \right\rangle = v_1^2 + v_2^2 \geq v_1^2 + v_2^2 - v_3^2 = \langle v, v \rangle. \tag{12.1}$$

Then $(d\tilde{x})_p$ is injective and this proves that $\tilde{x}$ is a local diffeomorphism, proving the result. □

We prove that a compact spacelike immersed surface is a graph provided its boundary projects onto a simple plane curve.

Proposition 12.1.7 *Let $x : M \to \mathbb{L}^3$ be a compact spacelike surface bounded by a simple closed curve Γ. Assume that there exists a spacelike plane P such that the orthogonal projection Γ' of Γ on P is a simple closed curve. Then there exists a diffeomorphism $F : \overline{\Omega} \to M$ from a bounded domain $\Omega \subset \mathbb{R}^2$ such that $x \circ F$ is a spacelike graph on $\overline{\Omega}$.*

Proof Assume without loss of generality that P is the horizontal plane $x_3 = 0$. Using (12.1), the projection $\tilde{x} = \pi \circ x : int(M) \to \mathbb{R}^2$ is a local diffeomorphism and, therefore, it is an open map. Let $\Omega = \tilde{x}(int(M)) \subset \mathbb{R}^2$, which is an open subset of $\mathbb{R}^2$, and let Ω' be the planar domain bounded by the plane simple closed curve $\Gamma' = \pi(\Gamma) = \tilde{x}(\partial M)$.

1. We prove $\partial\tilde{x}(M) \subset \Gamma'$. Since M is compact, for any $q \in \partial\tilde{x}(M)$ there exists a $p \in M$ such that $\tilde{x}(p) = q$. We show that $p \in \partial M$. Assuming the contrary, $p \in int(M)$ and there is an open neighborhood U_p of p in $int(M)$ and an open neighborhood V_q of q in Ω such that $\tilde{x} : U_p \to V_q$ is a diffeomorphism. This implies that $q \in \Omega$, contradicting that q is a boundary point of $\tilde{x}(M)$.
2. We prove $\Omega = \Omega'$. If there exists a point in Ω which is not in Ω' and since Ω is bounded, there are points in $\partial\Omega$ outside Ω', which is impossible. Analogously, if there is a point in Ω' which is not in Ω, there are points in $\partial\Omega$ inside Ω', which is again not possible.
3. As a consequence, $\tilde{x} : M \to \overline{\Omega}$ is a local diffeomorphism, and the compactness of M implies that $\tilde{x}$ is a covering map. Since $\overline{\Omega}$ is simply connected, the map $\tilde{x}$ must be a global diffeomorphism.

Hence, letting $F = \tilde{x}^{-1}$, we see that $x \circ F$ is the graph determined by the function $f = x_3 \circ F$. □

From Propositions 12.1.4 and 12.1.7, we have:

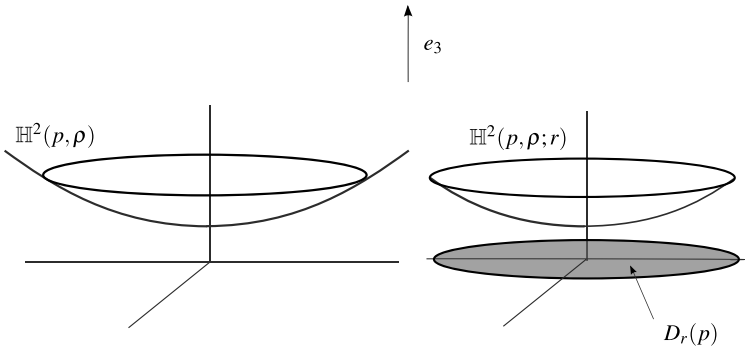

Fig. 12.1 *Left*: a hyperbolic plane. *Right*: a hyperbolic cap

Corollary 12.1.8 *A compact spacelike surface immersed in* $\mathbb{L}^3$ *spanning a plane simple closed curve is a graph.*

Umbilical surfaces in $\mathbb{L}^3$ are planes, hyperbolic planes and pseudospheres. Among them, the only spacelike surfaces are spacelike planes and hyperbolic planes. A *spacelike plane* is of the form as $P = \{x \in \mathbb{L}^3 : \langle x - p, v\rangle = 0\}$, where $p \in \mathbb{L}^3$ and v is a timelike vector. After an isometry of $\mathbb{L}^3$, a *hyperbolic plane*centered at p and radius $\rho > 0$ is of the form

$$\mathbb{H}^2(p,\rho) = \left\{x \in \mathbb{L}^3 : \langle x - p, x - p\rangle = -\rho^2, \langle x - p, e_3\rangle < 0\right\}.$$

See Fig. 12.1. This surface is isometric to the hyperbolic space of dimension 2. An orientation of $\mathbb{H}^2(p,\rho)$ is $N(x) = (x - p)/\rho$, and N points to the future direction. Let $\mathbb{H}^2 = \mathbb{H}^2(O, 1)$, O being the origin of $\mathbb{R}^3$:

$$\mathbb{H}^2 = \left\{(x_1, x_2, x_3) \in \mathbb{L}^3 : x_1^2 + x_2^2 - x_3^2 = -1, x_3 > 0\right\}.$$

If M is a spacelike surface of $\mathbb{L}^3$, the orientation N pointing to the future is a differentiable map $N : M \to \mathbb{H}^2$. This shows that in the context of spacelike surfaces of $\mathbb{L}^3$, the role of a sphere is played by $\mathbb{H}^2$.

Finally, we define a hyperbolic cap.

Definition 12.1.9 A hyperbolic cap of $\mathbb{L}^3$ is a compact piece of a hyperbolic plane whose boundary is a circle.

Indeed, after an isometry of $\mathbb{L}^3$, a hyperbolic cap is a compact surface that stems from the intersection of $\mathbb{H}^2(p,\rho)$ with a horizontal plane, as depicted in Fig. 12.1. For each $p \in \mathbb{L}^3$ and $r > 0$, consider the disk $D_r(p) = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - p_1)^2 + (x_2 - p_2)^2 \le r^2\}$. Define the hyperbolic cap $\mathbb{H}^2(p,\rho;r)$ as

$$\mathbb{H}^2(p,\rho;r) = \mathbb{H}^2(p,\rho) \cap \left\{x \in \mathbb{L}^3 : x_3 \le p_3 + \sqrt{r^2 + \rho^2}\right\}.$$

The hyperbolic cap $\mathbb{H}^2(p,\rho;r)$ is the graph of the function v_ρ

$$v_\rho : D_r(p) \to \mathbb{R}, \qquad v_\rho(x_1,x_2) = p_3 + \sqrt{\rho^2 + (x_1-p_1)^2 + (x_2-p_2)^2}$$

and the boundary condition is

$$v_\rho = p_3 + \sqrt{r^2+\rho^2} \quad \text{on } \partial D_r(p).$$

The boundary curve of $\mathbb{H}^2(p,\rho;r)$ is the circle of radius r given by the intersection of $\mathbb{H}^2(p,\rho;r)$ with the plane with equation $x_3 = p_3 + \sqrt{r^2+\rho^2}$. Note that the functions v_ρ are smooth on $D_r(p)$ for *any* $r>0$, in contrast to the case of Euclidean space. Recall that a hemisphere in $\mathbb{R}^3$ of mean curvature $H>0$ is the graph of a function u that is smooth in the interior of a circular disk of radius $1/H$ but u does not extend smoothly to the boundary.

12.2 The Mean Curvature of a Spacelike Surface

Let $x : M \to \mathbb{L}^3$ be a spacelike immersion of a surface M. Let $\mathfrak{X}(M)$ be the space of tangent vector fields to M and denote by ∇^0 the Levi-Civita connection of $\mathbb{L}^3$ and by ∇ the induced connection on M by the immersion x. The expression of the Gauss formula is

$$\nabla^0_X Y = \nabla_X Y + \sigma(X,Y), \quad X,Y \in \mathfrak{X}(M), \tag{12.2}$$

where σ is the second fundamental form. Let N be a unit normal vector field on M pointing to the future. If $X, Y \in \mathfrak{X}(M)$,

$$\langle A(X), Y\rangle = \langle \sigma(X,Y), N\rangle, \tag{12.3}$$

where A is the Weingarten endomorphism associated to N. As N is timelike, this relation implies

$$\sigma(X,Y) = -\langle \sigma(X,Y), N\rangle N = -\langle A(X), Y\rangle N, \tag{12.4}$$

and (12.2) now becomes

$$\nabla^0_X Y = \nabla_X Y - \langle A(X), Y\rangle N. \tag{12.5}$$

Definition 12.2.1 Let $x : M \to \mathbb{L}^3$ be a spacelike immersion of a surface M. The mean curvature vector field $\mathbf{H}$ and the Gauss curvature K of a spacelike surface are defined as

$$\mathbf{H} = \frac{1}{2}\text{trace}(\sigma), \qquad K = -\frac{\det(\sigma)}{\det(I)},$$

where $\mathrm{I} = \langle\cdot,\cdot\rangle$ is the first fundamental form of x. The mean curvature function H is defined as $\mathbf{H} = HN$, that is, $H = -\langle \mathbf{H}, N\rangle$.

In terms of the Weingarten endomorphism A (see also (12.4)), the curvatures H and K are

$$H = -\frac{1}{2}\text{trace}(A), \qquad K = -\det(A). \tag{12.6}$$

Since σ is symmetric, identity (12.3) yields $\langle A(X), Y\rangle = \langle X, A(Y)\rangle$. This means that $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a self-adjoint map with respect to the metric $\langle\cdot,\cdot\rangle$ and because the induced metric is positive definite, A is diagonalizable. Let κ_1 and κ_2 stand for the principal curvatures. Then (12.6) gives

$$H(p) = -\frac{\kappa_1(p) + \kappa_2(p)}{2}, \qquad K(p) = -\kappa_1(p)\kappa_2(p).$$

On the other hand,

$$0 \leq \left(\frac{\kappa_1 - \kappa_2}{2}\right)^2 = \left(\frac{\kappa_1 + \kappa_2}{2}\right)^2 - \kappa_1\kappa_2 = H^2 + K.$$

Thus $H^2 + K \geq 0$ and equality holds at a point p if and only if p is an umbilical point. Finally,

$$|\sigma|^2 = \sum_{i,j=1}^{2} \langle Ae_i, e_j\rangle^2 = \kappa_1^2 + \kappa_2^2 = 4H^2 + 2K.$$

As an example, for the hyperbolic plane $\mathbb{H}^2(p,\rho)$ with $N(x) = (x-p)/\rho$, we have $Av = -v/\rho$, $H = 1/\rho$ and $K = -1/\rho^2$.

Using a local parametrization

$$\Psi : U \subset \mathbb{R}^2 \to \mathbb{L}^3, \qquad \Psi = \Psi(x_1, x_2),$$

the expressions of H and K in terms of the coefficients of I and σ are

$$H = -\frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}, \qquad K = -\frac{eg - f^2}{EG - F^2}. \tag{12.7}$$

Let $u \in C^2(\Omega)$ be a function defined on a domain $\Omega \subset \mathbb{R}^2$ and consider the surface M given by $x_3 = u(x_1, x_2)$. Let $\Psi : \Omega \to \mathbb{L}^3$ denote the usual embedding $\Psi(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$. The coefficients of the first fundamental form are

$$E = 1 - u_{x_1}^2, \qquad F = -u_{x_1}u_{x_2}, \qquad G = 1 - u_{x_2}^2.$$

Thus $EG - F^2 = 1 - u_{x_1}^2 - u_{x_2}^2 = 1 - |\nabla u|^2$ and since the immersion is spacelike, $|\nabla u|^2 < 1$ on Ω. Take on M the orientation pointing to the future given by

$$N\big(x_1, x_2, u(x_1, x_2)\big) = \frac{(u_{x_1}, u_{x_2}, 1)}{\sqrt{1 - |\nabla u|^2}} = \frac{(\nabla u, 1)}{\sqrt{1 - |\nabla u|^2}}. \tag{12.8}$$

With respect to N, the mean curvature H satisfies

$$\left(1-u_{x_2}^2\right)u_{x_1x_1}+2u_{x_1}u_{x_2}u_{x_1x_2}+\left(1-u_{x_1}^2\right)u_{x_2x_2}=2H\left(1-|\nabla u|^2\right)^{\frac{3}{2}}, \tag{12.9}$$

that is,

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)=2H, \tag{12.10}$$

where div is the Euclidean divergence in $\mathbb{R}^2$. Observe that the mean curvature equation in Euclidean and Lorentz-Minkowski spaces can be expressed with a unified notation (compare with Eqs. (3.2) and (3.3)):

$$\left(1+\epsilon u_{x_2}^2\right)u_{x_1x_1}-2\epsilon u_{x_1}u_{x_2}u_{x_1x_2}+\left(1+\epsilon u_{x_1}^2\right)u_{x_2x_2}=2H\left(1+\epsilon|\nabla|^2\right)^{\frac{3}{2}}$$
$$\operatorname{div}\left(\frac{\nabla}{\sqrt{1+\epsilon|\nabla u|^2}}\right)=2H,$$

in the space $(\mathbb{R}^3,(dx_1)^2+(dx_2)^2+\epsilon(dx_3)^2)$, with $\epsilon=\pm1$. Again, the constant mean curvature equation (12.10) is of quasilinear elliptic type and we can apply the tangency principleas in Theorem 3.2.4. The same is true for the comparison principle and the proof given in Theorem 3.1.5 works in this context. As a consequence, the Dirichlet problem has a unique solution in the following sense:

Theorem 12.2.2 *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain. If M_1 and M_2 are two spacelike H-graphs on Ω with $\partial M_1=\partial M_2$, then $M_1=M_2$.*

A spacelike surface with zero mean curvature at all its points is called a *maximal surface*. This name is motivated by the fact that a maximal surface locally maximizes the surface area. Comparing with Proposition 2.1.8, we have:

Proposition 12.2.3 *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain and let M be a compact maximal graph on Ω. Then M maximizes the area among all graphs spanning ∂M.*

Proof We follow the same steps, defining the vector field $Y(x,y,z)=N(x,y,u(x,y))$, where N is given by (12.8). If M' is another graph on Ω with the same boundary as M, since $\operatorname{Div}(Y)=0$, we have

$$0=\int_M\langle Y,N\rangle\,dM+\int_{M'}\langle Y,N'\rangle\,dM'=-\operatorname{area}(M)+\int_{M'}\langle Y,N'\rangle\,dM'.$$

As N' and Y are unit timelike vectors pointing to the future, the Cauchy-Schwarz inequality says $\langle N',Y\rangle\le-1$. Thus

$$\operatorname{area}(M)=-\int_{M'}\langle Y,N'\rangle\,dM'\ge\int_{M'}dM'=\operatorname{area}(M').$$

□

12.3 Spacelike Surfaces with Non-empty Boundary

Given a simple closed curve $\Gamma \subset \mathbb{L}^3$, we investigate those spacelike cmc surfaces of $\mathbb{L}^3$ having Γ as its boundary. We may pose similar questions as in Euclidean ambient space such as if the symmetries of Γ carry to the surface or if there are restrictions on the possible values of mean curvature.

In order to illustrate these questions, we consider the simplest closed curve, Γ, namely a circle. First we need to make precise the notion of a circle in $\mathbb{L}^3$ because the metric is Lorentzian. See [LLS00] for further discussions. In the preceding section, we used the word 'circle' when we said that the boundary of $\mathbb{H}^2(p, \rho; r)$ was the circle

$$\partial\mathbb{H}^2(p, \rho; r) = \left\{x \in \mathbb{L}^3 : (x_1 - p_1)^2 + (x_2 - p_2)^2 = r^2, x_3 = p_3 + \sqrt{r^2 + \rho^2}\right\}. \tag{12.11}$$

This curve is a Euclidean circle in a horizontal plane and this plane is isometric to the Euclidean plane.

Definition 12.3.1 A spacelike circle in Lorentz-Minkowski space $\mathbb{L}^3$ is a curve included in a spacelike plane with non-zero constant curvature.

In the following, by a 'circle' we shall mean a spacelike circle. Recall that any curve in a spacelike plane P must be spacelike. Since P inherits a Riemannian metric, the notion of curvature for a curve included in P is defined in the usual sense. We may equivalently define a circle as the set of equidistant points of P from a fixed point of this plane, called its center. After a rigid motion of $\mathbb{L}^3$, we suppose that P is the plane $x_3 = 0$, and in this case, a circle in P may be viewed as a Euclidean circle. Here, we understand that a circle is always defined on its maximal domain and this implies that a circle of $\mathbb{L}^3$ is a closed curve. The parametrization is $\alpha(s) = p + r(\cos s, \sin s, 0)$, $s \in \mathbb{R}$, $p \in P$ and the curvature is $\kappa = 1/r$.

In the plane $x_3 = 0$, let Γ be a circle of radius $r > 0$ and centered at the origin. For each $H > 0$, there exists a hyperbolic cap with constant mean curvature H spanning Γ, say, $\mathbb{H}^2(-\sqrt{r^2 + \rho^2}e_3, \rho; r)$, $\rho = 1/H$. This surface is the hyperbolic cap $\mathbb{H}^2(O, \rho; r)$, followed by a vertical motion until its boundary lies contained in the plane $x_3 = 0$ and coincides with the circle Γ itself.

We call the attention to the fact that a circle contained in a hyperbolic plane separates the surface into two connected components, but only one of them is a compact set. In other words, when we intersect a hyperbolic plane with a spacelike plane, we only obtain one hyperbolic cap. This is a difference with Euclidean space, where *two* spherical caps are obtained by intersecting a sphere by a plane. Following our study of Euclidean and hyperbolic space, it is natural to ask:

> Are hyperbolic caps and planar disks the only compact spacelike cmc surfaces of $\mathbb{L}^3$ whose boundary is a circle?

Let Γ be a circle of radius $r > 0$. By Corollary 12.1.8, we know that a compact spacelike H-surface spanning Γ must be a graph. On the other hand, we already

know that for each $H \in \mathbb{R}$ there exists a planar disk or a hyperbolic cap spanning Γ and with constant mean curvature H. As a consequence of the uniqueness of spacelike cmc graphs given in Theorem 12.2.2, we conclude [ALT98]:

Corollary 12.3.2 *Planar disks and hyperbolic caps are the only compact spacelike cmc surfaces bounded by a circle.*

In the context of this book, this result contrasts to what happens in Euclidean and hyperbolic spaces because in this corollary there are no further assumptions on the surface, such as appeared in the three conjectures posed in $\mathbb{R}^3$ and $\mathbb{H}^3$. Only the case $|H| \leq 1$ in $\mathbb{H}^3$ presents the same conclusions (see Theorem 10.5.1).

Now, we consider a compact spacelike cmc surface whose boundary is a plane simple closed curve. In view of Corollary 12.1.8, we know that the surface is a graph and by a similar argument as in Corollary 3.1.6, the graph lies on one side of the plane P containing the boundary. Moreover, if P is the plane $x_3 = 0$ and if the graph M is future directed with $\partial M \subset P$, then $H > 0$ if and only if M lies below P.

We generalize this result in a more general setting. First, we drop the condition that H is constant and second, we consider that the boundary lies in a spacelike plane or a hyperbolic plane. We distinguish both cases. The proof of the next result has the same flavor as that of Corollary 3.1.6.

Theorem 12.3.3 *Let P be a spacelike plane. Let M be a compact surface and let $x : M \to \mathbb{L}^3$ be a spacelike immersion with $x(\partial M) \subset P$. If the mean curvature function H satisfies $H \neq 0$, then $x(int(M))$ lies on one side of P.*

Proof Without loss of generality, let P the plane $x_3 = 0$ and let $\{P_t : t \in \mathbb{R}\}$ be the foliation of $\mathbb{L}^3$ given by the horizontal planes $x_3 = t$. Consider on M the orientation N pointing to the future. By contradiction, assume that $int(M)$ has points on both sides of P. Then there are $p, q \in int(M)$ such that p (resp. q) is the highest (resp. lowest) point of M with respect to P. Suppose $x_3(q) \leq 0 < x_3(p)$ (the argument is similar if $x_3(q) < 0 \leq x_3(p)$). With respect to the chosen orientation on M, and because P_t is a minimal surface, $P_{x_3(p)} \geq M$ around p. The comparison principle gives $0 \geq H(p)$. Since $H \neq 0$ on M, $H < 0$ on M. At the lowest point q, the plane $P_{x_3(q)}$ is tangent to M at q and $M \geq P_{x_3(q)}$ around q (even if $x_3(q) = 0$). The comparison principle implies $H(q) \geq 0$, a contradiction. □

Corollary 12.3.4 *A compact maximal surface in $\mathbb{L}^3$ with planar boundary is a planar domain.*

Proof The result is a direct consequence of Proposition 12.1.7 and Theorem 12.2.2. However, we can do the proof without invoking the tangency principle, following the same ideas as the proof of Theorem 3.1.12. Let Γ be the boundary of M, which we assume is included in the plane P with equation $x_3 = 0$, and let D_r be a round disk of radius $r > 0$ such that $\Gamma \subset D_r$. Assume that M has points below P and let $M^- = M \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$ (similar arguments apply if M has points

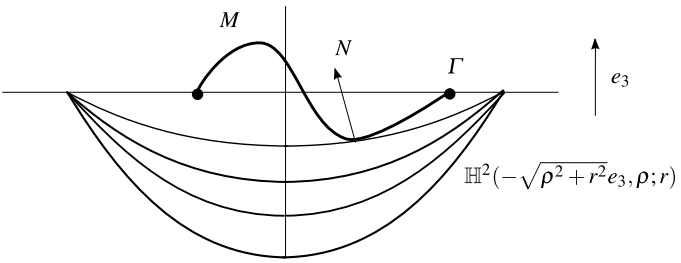

Fig. 12.2 Corollary 12.3.4: comparison of a maximal surface M with the hyperbolic caps $\mathbb{H}^2(-\sqrt{\rho^2+r^2}e_3,\rho;r)$

above P). Consider the 1-parameter family of hyperbolic caps with the same boundary ∂D_r given by

$$\left\{\mathbb{H}^2\left(-\sqrt{\rho^2+r^2}e_3,\rho;r\right):\rho>0\right\}.$$

Moving each of these surfaces down, and then back up, and since with the orientation pointing to the future, the mean curvature of the hyperbolic cap is positive, we obtain that M^- is included in the bounded 3-domain defined by $\mathbb{H}^2(-\sqrt{\rho^2+r^2}e_3,\rho;r)$ and D_r for all $\rho>0$. Note that the height with respect to P for each hyperbolic cap is $\sqrt{\rho^2+r^2}-\rho$. Letting $\rho\to+\infty$, the height goes to 0 and there would be a contact between some $\mathbb{H}^2(-\sqrt{\rho^2+r^2}e_3,\rho;r)$ and M^-, necessarily at an interior point: see Fig. 12.2. This contradicts the comparison principle. □

Assume now that the boundary curve lies in a hyperbolic plane. Given $H>0$, the space $\mathbb{L}^3$ is foliated by hyperbolic planes with mean curvature H. Indeed, given $\rho=1/H$, consider the family of hyperbolic planes $\{\mathbb{H}^2(te_3,\rho):t\in\mathbb{R}\}$ centered at te_3 and of radius ρ. In fact, $\mathbb{H}^2(te_3,\rho)$ is a vertical translations of $\mathbb{H}^2(O,\rho)$. All these surfaces have constant mean curvature H for the future-directed orientation and foliate the space $\mathbb{L}^3$ in the sense that two distinct hyperbolic planes of this family are disjoint and that for each point of $\mathbb{L}^3$ there is a unique hyperbolic plane through this point. The next result holds for non-constant mean curvature.

Theorem 12.3.5 *Let P be a hyperbolic plane of mean curvature $h>0$. Consider a compact surface M and let $x:M\to\mathbb{L}^3$ be a spacelike immersion with $x(\partial M)\subset P$. If $|H|\neq h$, then $x(int(M))$ lies on one side of P.*

Proof We may suppose that P is $\mathbb{H}^2(O,\rho)$, with $\rho=1/h$. The surface $\mathbb{H}^2(O,\rho)$ separates the space $\mathbb{L}^3$ into two domains, one of them is $W=\{x\in\mathbb{L}^3:\langle x,x\rangle<-\rho^2,x_3>0\}$, the so-called mean convex side of $\mathbb{H}^2(O,\rho)$, because it is the domain towards which the orientation that makes $h>0$ points. Let us suppose for contradiction that $int(M)\cap(\mathbb{L}^3\setminus W)\neq\emptyset$ and $int(M)\cap W\neq\emptyset$ (the argument is similar if $int(M)\cap(\mathbb{L}^3\setminus\overline{W})\neq\emptyset$ and $int(M)\cap\overline{W}\neq\emptyset$). We orient M by the future-directed orientation. By the compactness of M, for values t close enough to $-\infty$, $\mathbb{H}^2(te_3,\rho)$

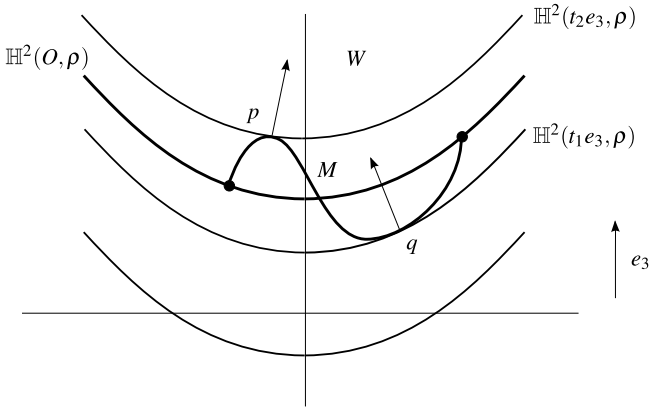

Fig. 12.3 The contradiction in the proof of Theorem 12.3.5

does not intersect M. Let $t \nearrow 0$ until $\mathbb{H}^2(te_3, \rho)$ first touches M at $t = t_1 \leq 0$ at an interior point q. This occurs because $int(M) \cap (\mathbb{L}^3 \setminus W) \neq \emptyset$. First observe that M and $\mathbb{H}^2(t_1e_3, \rho)$ are oriented to the future and thus both orientations agree at q as depicted in Fig. 12.3. The comparison principle says that $H(q) \geq h$.

Consider now $t > 0$. Approaching from $t = +\infty$, we continue until $\mathbb{H}^2(te_3, \rho)$ first touches M at $t = t_2 > 0$. Here t_2 is positive because $int(M) \cap W \neq \emptyset$. Let $p \in \mathbb{H}^2(t_2e_3, \rho) \cap int(M)$. Now, $\mathbb{H}^2(t_2e_3, \rho) \geq M$ around p and the comparison principle yields $h \geq H(p)$. By connectedness, there exists an $m \in M$ with $H(m) = h$, which is a contradiction. □

In the case that $|H| = h$, the same argument proves:

Corollary 12.3.6 *Let P be a hyperbolic plane of mean curvature $h > 0$. Let M be a compact surface and let $x : M \to \mathbb{L}^3$ be a spacelike immersion with $x(\partial M) \subset P$. If $|H| = h$, then $x(M) \subset P$ or $x(int(M))$ lies in the mean convex side of P.*

12.4 The Flux Formula

We turn to the study of compact spacelike cmc surfaces with circular boundary. In this section we provide a new proof of Corollary 12.3.2 by means of the flux formula. This is surprising when compared with Euclidean and hyperbolic space, where the flux formula does not allow us conclude that the surface is umbilical. As in Euclidean space, the flux formula relates two integrals along the boundary, one of which depends only on H and the boundary curve. Using the same 1-form as in the proof of (5.1), we obtain:

Theorem 12.4.1 (Flux formula) *Let M be a compact surface and let $x : M \to \mathbb{L}^3$ be a spacelike immersion spanning a simple closed curve Γ. Assume that H is*

constant. For any fixed vector $a \in \mathbb{L}^3$ we have

$$H \int_{\partial M} \langle \alpha(s) \times \alpha'(s), a \rangle ds - \int_{\partial M} \langle N(s) \times \alpha'(s), a \rangle ds = 0, \tag{12.12}$$

where $\alpha : \partial M \to \Gamma$ is a parametrization of Γ, N an orientation on M and $\times$ is the Lorentzian cross product.

Proof We only remark that, using the same notation as in Theorem 5.1.1, the basis $\{e_1, e_2, N\}$ is positively oriented is equivalent to $\det(e_1, e_2, N) = 1$ and $e_1 \times e_2 = -N$. □

Assume now that Γ is a simple closed spacelike curve contained in a plane P, which is necessarily spacelike. Let $P = \langle a \rangle^{\perp}$ where a is a unit timelike vector oriented to the future. Let us consider M oriented to the future direction by N. As in Euclidean space, we obtain the corresponding formula (5.5), which is

$$-2 \int_M \langle N, a \rangle \, dM = \int_{\partial M} \langle \alpha(s) \times \alpha'(s), a \rangle ds, \tag{12.13}$$

where ∂M is positively oriented. In the closed topological surface $M \cup \Omega$, where $\Omega \subset P$ is the domain bounded by Γ, we extend the orientation given by N on M and let η be the induced orientation on Ω. As $\eta = -a$, we have

$$\int_M \langle N, a \rangle \, dM = -\int_{\Omega} \langle \eta, a \rangle \, d\Omega = -\operatorname{area}(\Omega).$$

Then (12.12) and (12.13) give

$$\int_{\partial M} \langle N(s) \times \alpha'(s), a \rangle ds = 2H \operatorname{area}(\Omega). \tag{12.14}$$

It is worth pointing out that, in contrast to the Euclidean case, formula (12.14) does not imply any restriction on the possible values of the constant mean curvature. This is because it is not possible to get an upper estimate of the function $\langle N \times \alpha', a \rangle$ since $N \times \alpha'$ is a spacelike vector and a is timelike. For example, given a circle Γ of radius $r > 0$, the hyperbolic caps $\mathbb{H}^2(O, \rho; r)$, $0 < \rho < \infty$, followed by appropriate vertical translations, describe a family of compact spacelike H-surfaces spanning Γ where $H = 1/\rho$ varies in the interval $(0, \infty)$.

The next proposition gives the Lorentzian versions of Eqs. (2.22) and (2.24). We omit the proofs.

Proposition 12.4.2 *Let $x : M \to \mathbb{L}^3$ be a spacelike immersion and let $a \in \mathbb{L}^3$. Then*

$$\Delta \langle x, a \rangle = 2H \langle N, a \rangle. \tag{12.15}$$

If the immersion has constant mean curvature, then

$$\Delta \langle N, a \rangle - |\sigma|^2 \langle N, a \rangle = 0. \tag{12.16}$$

We are in a position to prove Corollary 12.3.2 using the flux formula [ALT98].

Theorem 12.4.3 *Planar disks and hyperbolic caps are the only compact spacelike cmc surfaces bounded by a circle.*

Proof After an isometry of $\mathbb{L}^3$, assume that the boundary Γ is a circle of radius $r > 0$ in the plane P with equation $x_3 = 0$. Let M be a compact spacelike H-surface spanning Γ. If $H = 0$, we know from Corollary 12.3.4 that the surface is a planar disk. Now assume that $H \neq 0$ and orient M by the future-directed orientation N. Equations (12.15) and (12.16) give

$$\Delta\big(\langle N, e_3\rangle - H\langle x, e_3\rangle\big) = 2\big(H^2 + K\big)\langle N, e_3\rangle. \tag{12.17}$$

By the divergence theorem,

$$\int_{\partial M}\big(-\langle dN(v), e_3\rangle + H\langle v, e_3\rangle\big)\,ds = 2\int_M \big(H^2 + K\big)\langle N, e_3\rangle\,dM \leq 0,$$

where ν is the inward unit conormal vector of M along ∂M and where we have used $H^2 + K \geq 0$ and $\langle N, e_3\rangle \leq -1$. Thus

$$\int_{\partial M}\big(\langle A\nu, \nu\rangle + H\big)\langle \nu, e_3\rangle\,ds \leq 0.$$

As $2H = -\langle A\alpha', \alpha'\rangle - \langle A\nu, \nu\rangle$, we see that

$$-\int_{\partial M}\langle A\alpha', \alpha'\rangle\langle \nu, e_3\rangle\,ds \leq H\int_{\partial M}\langle \nu, e_3\rangle. \tag{12.18}$$

If α is an arc length parametrization, then $\alpha''(s) = -\alpha(s)/r^2$. As $N(s) \times \alpha'(s) = \nu(s)$ and e_3 is the future orientation of the plane P, the flux formula (12.14) can be expressed as

$$\int_{\partial M}\langle \nu, e_3\rangle\,ds = 2\pi H r^2. \tag{12.19}$$

Moreover,

$$\langle A\alpha', \alpha'\rangle = -\langle N', \alpha'\rangle = \langle N, \alpha''\rangle = -\frac{1}{r^2}\langle N, \alpha\rangle,$$
$$\langle N, \alpha\rangle = -\langle \alpha' \times \nu, \alpha\rangle = -\langle \alpha \times \alpha', \nu\rangle = r\langle \nu, e_3\rangle.$$

Then

$$\langle A\alpha'(s), \alpha'(s)\rangle = -\frac{1}{r}\langle \nu(s), e_3\rangle.$$

Substituting into (12.18) and using (12.19), we find that

$$\int_{\partial M}\langle \nu, e_3\rangle^2\,ds \leq 2\pi H^2 r^3. \tag{12.20}$$

Finally, flux formula (12.14) and the Cauchy-Schwarz inequality imply

$$4\pi^2 H^2 r^4 = \left(\int_{\partial M} \langle \nu, e_3\rangle \, ds\right)^2 \leq 2\pi r \int_{\partial M} \langle \nu, e_3\rangle^2 \, ds. \tag{12.21}$$

Combining (12.20) and (12.21), we have equalities and so,

$$\int_{\partial M} \langle \nu, e_3\rangle^2 \, ds = 2\pi H^2 r^3.$$

In particular, $H^2 + K = 0$ on M, which means that M is umbilical. As $H \neq 0$, the surface is a hyperbolic cap because Γ is a circle. □

Remark 12.4.4 The proof is similar to that of Theorem 2.3.8. In this case, since the surface is spacelike, the function $\langle N, e_3\rangle$ does not vanish on M. In Theorem 2.3.8 we assumed that the surface is a graph, which means that $\langle N, e_3\rangle$ does not vanish.

We point out that in Euclidean space, Eqs. (2.22) and (2.24) allowed us to obtain estimates of the height of a cmc graph or an embedded cmc surface (Theorem 2.3.5). In Lorentz-Minkowski space, it is not possible to get these estimates depending only on H. For this, we follow the same argument assuming that the boundary of the surface lies in the horizontal plane $x_3 = 0$. Let N be the orientation of the surface pointing to the future. In view of (12.17), $\Delta(\langle N, e_3\rangle - H\langle x, e_3\rangle) \leq 0$ and the maximum principle gives

$$\langle N, e_3\rangle - H\langle x, e_3\rangle \geq \min_{\partial M}\big(\langle N, e_3\rangle - H\langle x, e_3\rangle\big) = \min_{\partial M}\langle N, e_3\rangle.$$

On the other hand,

$$\min_{\partial M}\langle N, e_3\rangle \leq -1,$$

which is precisely the reverse inequality and we cannot follow the same reasoning. Indeed, let $\mathbb{H}^2(O, \rho)$ be the hyperbolic plane of mean curvature $H = 1/\rho$ and fix a horizontal plane Π. Intersect $\mathbb{H}^2(O, \rho)$ with a horizontal plane obtaining hyperbolic caps and next displace these caps vertically until the boundaries lie in Π. Then we obtain H-surfaces with arbitrary height spanning circles contained in Π.

However, motivated by the above examples, we get height estimates for a compact spacelike cmc surface in terms of H and the diameter $\delta(\Omega)$ of Ω, the latter being the domain where the surface is a graph.

Theorem 12.4.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a function whose graph is a spacelike surface with mean curvature function H. Suppose $h > 0$ such that $|H| \leq h$. Then*

$$\min_{\partial\Omega} u - \frac{1}{h}\left(\sqrt{1 + \frac{\delta(\Omega)^2 h^2}{4}} - 1\right) \leq u \leq \max_{\partial\Omega} u + \frac{1}{h}\left(\sqrt{1 + \frac{\delta(\Omega)^2 h^2}{4}} - 1\right). \tag{12.22}$$

In the particular case that $H = 0$, we have $\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u$.

Proof The estimates are obtained by comparing the graph of u with hyperbolic caps with mean curvature h coming from below and above. There is no loss of generality in assuming that Ω is included in the closed disk $D_r(0)$ of the plane P of equation $x_3 = 0$ where $r = \delta(\Omega)/2$. Let v_ρ denote the function whose graph is the hyperbolic cap $\mathbb{H}^2(O, \rho; r)$, $\rho = 1/h$. Let M be the graph of u. Translate $\mathbb{H}^2(O, \rho; r)$ vertically downward until it is disjoint from M. Then move $\mathbb{H}^2(O, \rho; r)$ up until $\mathbb{H}^2(te_3, \rho; r)$ first touches M at $t = t_1$. Denote by p the contact point. The mean curvature of this hyperbolic cap is h with $h \geq |H|$. By the tangency principle and $\Omega \subset D_r(0)$, there cannot exist touching points at a time t_1 until the boundary of $\mathbb{H}^2(t_1 e_3, \rho; r)$ arrives at the position $\min_{\partial\Omega} u$. Then

$$t_1 + \sqrt{\rho^2 + r^2} \leq \min_{\partial\Omega} u. \tag{12.23}$$

There are two possibilities for the point p. If p is an (interior or boundary) tangent point between M and $\mathbb{H}^2(t_1 e_3, \rho; r)$ and because $|H| \leq h$, the tangency principle tells us that $|H| = h$, $M \subset \mathbb{H}^2(t_1 e_3, \rho; r)$ and we have equality in (12.23) and consequently in (12.22).

The other possibility is that the point p is a boundary point where both surfaces M and $\mathbb{H}^2(t_1 e_3, \rho; r)$ are not tangent. Then we have equality in (12.23), that is, $t_1 + \sqrt{\rho^2 + r^2} = \min_{\partial\Omega} u$. If $q \in \Omega$, we find that

$$u(q) \geq v_\rho(q) \geq v_\rho(0) = t_1 + \rho = \min_{\partial\Omega} u + \rho - \sqrt{r^2 + \rho^2}.$$

Since $r = \delta(\Omega)/2$ and $\rho = 1/h$, we derive the estimate of the left-hand side of (12.22). In order to prove the right-hand inequality in (12.22), we use a similar reasoning with the reflection of $\mathbb{H}^2(te_3, \rho; r)$ about a horizontal plane, coming from above.

For the case $H = 0$, the proof can be achieved by letting $h \to 0$ in (12.22) or by comparing M with horizontal planes. □

If the mean curvature is constant and the boundary is included in the plane P, Theorem 12.3.3 gives:

Corollary 12.4.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let $H \neq 0$ be a real number and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of (12.9) with $u = 0$ on $\partial\Omega$. Then*

$$0 < |u| \leq \frac{1}{|H|}\left(\sqrt{1 + \frac{\delta(\Omega)^2 H^2}{4}} - 1\right) \quad \text{on } \Omega$$

and equality holds if and only if the graph of u describes a hyperbolic cap.

Remark 12.4.7 Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ describe a spacelike graph whose mean curvature is not necessarily constant. If Ω is a bounded domain, then u is bounded in terms of Ω and the value of u on $\partial\Omega$. Indeed, let $p \in \Omega$ and consider $p_0 \in \partial\Omega$

the nearest boundary point to p. In particular, the open segment joining p and p_0 is included in Ω. As $|\nabla u| < 1$, the function u is Lipschitz and so,

$$\begin{aligned} \big|u(p)\big| &\leq \big|u(p_0)\big| + \big|u(p) - u(p_0)\big| \leq \max_{\partial\Omega} |u| + |p - p_0| \\ &\leq \max_{\partial\Omega} |u| + \frac{\delta(\Omega)}{2}. \end{aligned} \tag{12.24}$$

However, if the mean curvature is a constant $H \neq 0$, the estimate (12.22) improves (12.24).

We may obtain similar height estimates by comparing the graph M with a spacelike cylinder. After a rigid motion of $\mathbb{L}^3$, a *spacelike cylinder* of radius $r > 0$ is

$$C_r = \big\{(x_1, x_2, x_3) \in \mathbb{L}^3 : x_2^2 - x_3^2 = -r^2, x_3 > 0\big\}.$$

Take the Gauss map $N(x_1, x_2, x_3) = (0, x_2, x_3)/r$, which is future-directed. Then the mean curvature is $H = 1/(2r)$. Given a bounded domain $\Omega \subset \mathbb{R}^2$, consider the set A of all pairs (L_1, L_2), where $L_1, L_2 \subset \mathbb{R}^2$ are two parallel lines such that Ω is included in the strip of $\mathbb{R}^2$ determined by L_1 and L_2. As $\overline{\Omega}$ is compact, let

$$\Theta = \min\big\{\mathrm{dist}(L_1, L_2) : (L_1, L_2) \in A\big\}.$$

Observe that $\Theta \leq \delta(\Omega)$. After a rigid motion if necessary, assume that Ω is included in the strip $\{(x_1, x_2) \in \mathbb{R}^2 : -\Theta/2 < x_2 < \Theta/2\}$. Consider the cylinder C_r, $r = 1/(2h)$. Moving C_r vertically, as in the proof of Theorem 12.4.5 with hyperbolic caps, we obtain:

Corollary 12.4.8 *Let $\Omega \subset \mathbb{R}^2$ be a compact domain and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a function whose graph is a spacelike surface with mean curvature function H. Suppose $h > 0$ such that $|H| \leq h$. Then*

$$\min_{\partial\Omega} u - \frac{1}{2h}\big(\sqrt{1 + \Theta^2 h^2} - 1\big) \leq u \leq \max_{\partial\Omega} u + \frac{1}{2h}\big(\sqrt{1 + \Theta^2 h^2} - 1\big). \tag{12.25}$$

Notice that this estimate cannot be compared with (12.22) because $\Theta \leq \delta(\Omega)$.

12.5 A Monotonicity Formula for the Area

From the preceding section, we know that there are no height estimates for spacelike cmc surfaces depending only on the mean curvature. However, we will obtain height estimates that relate to the area of the surface, as was established for Euclidean space for Theorem 6.1.1 [Lop04].

Theorem 12.5.1 *Let $x : M \to \mathbb{L}^3$ be a spacelike immersion of a compact surface with boundary included in a plane P. Assume the mean curvature H is constant. If h^+ denotes the height of M above P, we have*

$$h^+ \leq \frac{|H|A^+}{2\pi}, \tag{12.26}$$

where A^+ is the area of the region of M above the plane P. Equality holds if and only if M is a planar domain ($H = 0$) or a hyperbolic cap ($H \neq 0$).

Proof After an isometry of the ambient space, assume that P is the plane $x_3 = 0$. Let $\Gamma = x(\partial M)$. If $H = 0$, M is included in P by the tangency principle, proving the result.

Let us assume then $H \neq 0$ and that there are points of M above P. Consider the function $f : M \to \mathbb{R}$ given by $f = -\langle x, e_3\rangle = x_3$, which measures the height of M with respect to the plane P. Let $h^+ = \max_{p\in M} f(p) \geq 0$ and define

$$M(t) = \big\{p \in M : f(p) \geq t\big\}, \qquad \Gamma(t) = \big\{p \in M : f(p) = t\big\}.$$

Set $A(t)$ and $L(t)$ to be equal to the area and the length of $M(t)$ and $\Gamma(t)$ respectively.

We compute the critical points of the function f. Since $\nabla\langle x, e_3\rangle = e_3 + \langle N, e_3\rangle N$, $p \in M$ is a critical point of f if $|\nabla f|^2(p) = -1 + \langle N(p), e_3\rangle^2 = 0$, i.e., $N(p) = e_3$. Define on M the functions $g_i = \langle N, e_i\rangle$, $i = 1, 2$, where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Therefore the set $\mathcal{C}$ of critical points of f is contained in $\mathcal{N}_1 \cap \mathcal{N}_2$, where $\mathcal{N}_i = g_i^{-1}(\{0\})$ is the nodal line of g_i, $i = 1, 2$. As in Theorem 6.1.1, the set $\mathcal{C}$ has zero measure. It follows that $A(t)$ is a continuous function and the coarea formulaimplies

$$A'(t) = -\int_{\Gamma(t)} \frac{1}{|\nabla f|}\, ds_t,$$

where ds_t is the line element on the level $\Gamma(t)$. By Hölder's inequality, we find that

$$L(t)^2 \leq \int_{\Gamma(t)} \frac{1}{|\nabla f|}\, ds_t \int_{\Gamma(t)} |\nabla f|\, ds_t = -A'(t) \int_{\Gamma(t)} |\nabla f|\, ds_t. \tag{12.27}$$

Recall that $|\nabla f|$ along the curve $\Gamma(t)$ is $|\nabla f|^2 = \langle \nu_t, e_3\rangle^2$, where ν_t is the inner unit conormal vector of $M(t)$ along $\Gamma(t)$. As $M(t)$ lies above the plane $P_t = \{x \in \mathbb{L}^3 : -\langle x, e_3\rangle = t\}$, we know $\langle \nu_t, e_3\rangle \leq 0$. Hence

$$|\nabla f|_{|\Gamma(t)} = -\langle \nu_t, e_3\rangle.$$

It follows from (12.27) that

$$L(t)^2 \leq A'(t) \int_{\Gamma(t)} \langle \nu_t, e_3\rangle\, ds_t. \tag{12.28}$$

For almost $t \geq 0$, and by the flux formula (12.14) we have

$$-\int_{\partial M(t)} \langle \nu_t, e_3 \rangle \, ds_t = 2|H| a(\Omega_t),$$

where $\Omega_t \subset P_t$ is the region defined by $\Gamma(t)$. Then (12.28) becomes

$$L(t)^2 \leq -2|H| A'(t) a(\Omega_t). \tag{12.29}$$

A similar argument as in Theorem 6.1.1 gives

$$2\pi \leq -|H| A'(t) \tag{12.30}$$

for all $t \geq 0$. Integrating this inequality from 0 to the height h^+, we get

$$2\pi h^+ \leq |H| \big(A(0) - A(h)\big) = |H| A^+,$$

yielding the desired estimate (12.26).

Now we suppose that (12.26) is an equality. In such case, we have equality in (12.27) and in the isoperimetric inequalities. This means that $|\nabla f|$ is a constant function in each $\Gamma(t)$ and $\Gamma(t)$ is a circle, for all $t > 0$. By Theorem 12.4.3, the surface is a hyperbolic cap. On the other hand, it is straightforward that for a planar domain and a hyperbolic cap, (12.26) is an equality, and this concludes the proof of Theorem 12.5.1. □

Corollary 12.5.2 *A compact spacelike surface M of $\mathbb{L}^3$ with constant mean curvature and with planar boundary has area greater than that of a hyperbolic cap with the same height and mean curvature as M.*

12.6 The Dirichlet Problem

Let $\Omega \subset \mathbb{R}^2$ be a smooth domain, $H \in \mathbb{R}$ and $\varphi \in C^0(\partial\Omega)$. In this section we investigate the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = 2H & \text{in } \Omega \\ |\nabla u| < 1 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{12.31}$$

Here div and ∇ are the Euclidean divergence and gradient operators of $\mathbb{R}^2$. If u is a solution of this problem, its graph is a spacelike surface with constant mean curvature H with respect to the orientation given in (12.8) and the boundary curve is the graph of the function φ. The second condition in (12.31) means that the surface is spacelike, as was shown in Sect. 12.2. Let $P(H, \varphi)$ denote the problem (12.31).

The study of the Dirichlet problem (12.31) presents analogies with the Euclidean case, but also some differences. Among these, we mention the following:

1. We consider the problem of comparing the size of Ω with the value H. Given $H \neq 0$, hyperbolic caps are H-graphs defined in arbitrarily large domains. This contrasts with Theorem 3.3.2.
2. Let $H \neq 0$. There are spacelike H-surfaces that are graphs on $\mathbb{R}^2$ such as, for example, the hyperbolic planes $z = \sqrt{x^2 + y^2 + 1/H^2}$.
3. As in the Euclidean version, Bernstein's theorem holds in this setting and was proved by S.Y. Cheng and S.T. Yau [ChYa76]: the only maximal graph defined on $\mathbb{R}^2$ is a spacelike plane. However, the result holds for arbitrary dimensions, that is, for maximal hypersurfaces in $\mathbb{L}^n$. In contrast, a famous theorem of E. Bombieri, E. De Giorgi and E. Giusti asserts that in Euclidean space, Bernstein's theorem is only valid for minimal hypersurfaces of $\mathbb{R}^n$ with $n \leq 7$ [BGG].

In order to illustrate the techniques involved in the study of the Dirichlet problem, and to compare them with the Euclidean case, we shall restrict our efforts to the case that Ω is bounded (see [Maz07b] for maximal graphs in unbounded domains).

We employ the method of continuitydefining the set

$$\mathcal{S} = \left\{ t \in [0,1] : \text{there exists a solution } u_t \text{ of } P(tH, t\varphi) \right\}.$$

First, we see that $\mathcal{S} \neq \emptyset$ because the domain Ω itself is a graph and so, $0 \in \mathcal{S}$. Next we prove that the set $\mathcal{S}$ is open in $[0,1]$ using the implicit function theorem. The linearized operator of the mean curvature is the Jacobi operator $L = \Delta - |\sigma|^2$ (see (12.16)). In this context, the function $\langle N, e_3 \rangle$, where N is the Gauss map, satisfies $\langle N, e_3 \rangle < 0$ and $L(\langle N, e_3 \rangle) = 0$. By Remark 8.1.5, this suffices to show that the kernel of L is trivial and $\mathcal{S}$ is an open set of $[0,1]$.

Finally, we show that $\mathcal{S}$ is closed in $[0,1]$. This is equivalent to having C^1-estimates of the solutions u_t for $t \in \mathcal{S}$. The C^0-estimates are guaranteed by Corollary 12.4.6. Moreover, when $\varphi = 0$, if u is a solution of $P(H, 0)$, then $u \leq u_t \leq 0$ for all $t \in [0,1]$ and the estimate of $|u|$ implies that of $|u_t|$.

Estimates of $|\nabla u|$are obtained by means of the function $\langle N, e_3 \rangle$ and Eq. (12.16). As $\Delta \langle N, e_3 \rangle = |\sigma|^2 \langle N, e_3 \rangle \leq 0$, we have $\inf_M \langle N, e_3 \rangle = \min_{\partial M} \langle N, e_3 \rangle$. Thus

$$\sup_{\overline{\Omega}} |\nabla u_t| = \max_{\partial \Omega} |\nabla u_t|.$$

By the spacelike condition $|\nabla u_t| < 1$ and the previous argument, we deduce that there are gradient estimates provided there is a constant $C = C(H, \Omega) < 1$ depending only on H and Ω such that

$$\max_{\partial \Omega} |\nabla u_t| \leq C. \tag{12.32}$$

This guarantees the boundedness of the denominator in the mean curvature equation (12.31). If $\varphi = 0$ and in view of Lemma 8.1.12 and since $u \leq u_t \leq 0$, we have

$$\max_{\partial \Omega} |\nabla u_t| \leq \max_{\partial \Omega} |\nabla u|.$$

Remark 12.6.1 For Euclidean graphs, the smoothness of the solution on $\partial\Omega$ was guaranteed if the graph close to the boundary point did not blow-up at infinity. For a spacelike graph, we have to prevent the possibility that $|\nabla u| \to 1$ as $(x, y) \to \partial\Omega$. The existence of the constant C shows that the surface cannot 'go null' in the terminology of Marsden and Tipler [MT80, p. 124].

Now we are in position to state the main theorem of this section, due to R. Bartnik and L. Simon [BS82], which says that given a bounded domain, there exists a solution of the corresponding Dirichlet problem $P(H, \varphi)$ for *any* H and for *any* spacelike function φ.

Theorem 12.6.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Given $H \in \mathbb{R}$ and a spacelike function $\varphi \in C^0(\partial\Omega)$, there is a solution of $P(H, \varphi)$.*

Other existence results appear in [Bay03, Gri00, Kly95, Qui85, Tho12, Tre82, Urs03].

For simplicity, the proof will be given in the case $\varphi = 0$ and we will outline the proof of the general case. Before proving the theorem, we will study the particular case that Ω is *convex*. The proof that we present in this case is different from [BS82]; it is similar to the proof of Theorem 8.3.2, using on this occasion spacelike cylinders as barriers.

Proposition 12.6.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Given $H \in \mathbb{R}$, there is a solution of $P(H, 0)$.*

Proof Let u be a solution of the Dirichlet problem $P(H, 0)$. Let C_r be the cylinder $x_2^2 - x_3^2 = -r^2, x_3 > 0$ whose mean curvature is $H = 1/(2r)$. For each $m > r$, denote by $C_r(m)$ the piece of C_r given by

$$C_r(m) = \big\{(x_1, x_2, x_3) \in C_r : r \leq x_3 \leq m\big\}.$$

This surface is a graph on the strip $\Omega_{r,m} = \{(x_1, x_2) \in \mathbb{R}^2 : -\sqrt{m^2 - r^2} \leq x_2 \leq \sqrt{m^2 - r^2}\}$. See Fig. 12.4.

There is no loss of generality in assuming that $H > 0$ and letting M be the graph of u. In particular, $u < 0$ on Ω. Fix $r = 1/(2H)$ and take m sufficiently large so that $\delta(\Omega) < 2\sqrt{m^2 - r^2}$ and $\Omega \subset \Omega_{r,m}$. By vertical translations ϕ_t, we move $C_r(m)$ down until $\phi_t(C_r(m))$ does not intersect M and next, we increase t until $\phi_t(C_r(m))$ first intersects M at $t = t_1$. Since M and $\phi_{t_1}(C_r(m))$ are oriented pointing to the future and $\Omega \subset \Omega_{r,m}$, the tangency principle prohibits a tangent point between both surfaces. Thus the value t_1 is given by the property that $\partial\phi_{t_1}(C_r(m))$ is included in the plane $x_3 = 0$, that is, $t_1 = -m$. Furthermore, M lies in the 3-domain bounded by $\phi_{t_1}(C_r(m)) \cup \Omega_{r,m}$.

We move the surface $\phi_{t_1}(C_r(m))$ horizontally until it touches M. Again, the tangency principle implies that we can move $\phi_{t_1}(C_r(m))$ without contacting M until its boundary touches $\partial\Omega$. Since the boundary of $\Omega_{r,m}$ is formed by two parallel lines,

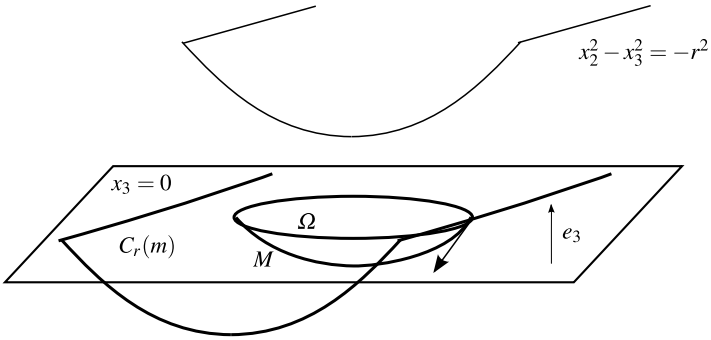

Fig. 12.4 Comparing cylinders with spacelike graphs with planar boundary. Along $\partial\Omega$, the slope of the graph is less than that of the cylinder

$\delta(\Omega) < 2\sqrt{m^2 - r^2}$ and Ω is a convex domain, the above displacements can be performed in any horizontal direction and touching each point of $\partial\Omega$. In conclusion we deduce that

$$\sup_{\partial\Omega} |\nabla u| \leq \sup_{\partial\Omega} |\nabla w|$$

along $\partial\Omega$, where w is the function defining the cylinder $\phi_{t_1}(C_r(m))$. This function w is $w(x_1, x_2) = \sqrt{x_2^2 + r^2} - m$. On $\partial\Omega_{r,m}$ we find that

$$|\nabla w| = \frac{\sqrt{m^2 - r^2}}{m} = C < 1$$

and this constant C depends only on m and r, or in other words, on Ω and H, proving the result. □

We return to the proof of Theorem 12.6.2. Imitating the Euclidean setting we need to understand the behaviour of the rotational spacelike cmc surfaces of $\mathbb{L}^3$. Consider the rotational surface about the x_3-axis obtained by the curve $(r, 0, w(r))$, $0 \leq a < r < b$. With respect to the orientation pointing to the future (12.8), the mean curvature H satisfies

$$2H = \frac{w''}{(1 - w'^2)^{3/2}} + \frac{w'}{r\sqrt{1 - w'^2}}. \tag{12.33}$$

Recall that the spacelike condition is equivalent to $1 - w'^2 > 0$. Multiplying by r, we derive a first integral, namely,

$$Hr^2 + c = \frac{rw'}{\sqrt{1 - w'^2}}$$

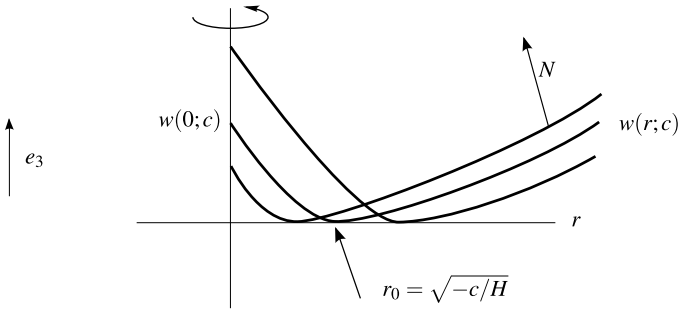

Fig. 12.5 Let $H > 0$. Rotational spacelike H-surfaces $\{S(c) : c < 0\}$ parameterized by c. The rotation curve is $w = w(r; c)$ where $w(r_0) = 0$ with $r_0 = \sqrt{-c/H}$

for a constant $c \in \mathbb{R}$. This is written as

$$w' = \frac{Hr^2 + c}{\sqrt{r^2 + (Hr^2 + c)^2}}. \tag{12.34}$$

Take $H > 0$. If $c = 0$, the solution is $w(r) = \sqrt{1/H^2 + t^2} + w_0$, for a constant w_0. In this case, the surface describes, up a vertical translation, a hyperbolic plane of type $\mathbb{H}^2(O, 1/H)$.

Assume $c \neq 0$ and denote by $S(c)$ the rotational surface. Given $r_0 > 0$, consider the initial condition

$$w(r_0) = w_0 > 0.$$

Denote the solution by $w = w(r; c)$ to indicate the dependence on c, which is defined on an interval around r_0. As the function w' is bounded, w is defined as long as (12.33) is valid, that is, $r > 0$ and $w'^2 < 1$. This means that the maximal interval of w is $(0, \infty)$. Assume that the integration constant c is negative. Let $r_0 = \sqrt{-c/H}$ and write $w = w(r; c)$ to indicate the dependence on c. It is straightforward that the curves $w(r; c)$ have the following properties. See Fig. 12.5.

1. The function w has at $r = r_0$ an absolute minimum, $w(r_0) =$ and $w'' > 0$ on $(0, \infty)$. Moreover, $\lim_{c\to-\infty} r_0(c) = +\infty$.
2. $\lim_{r\to 0} w'(r; c) = -1$. This means that $S(c)$ presents a singularity at the intersection point with the rotation axis and at this point, the surface is tangent to the (backward) light-cone from $w(0; c)$.
3. $\lim_{c\to-\infty} w(0; c) = +\infty$.

Proof of Theorem 12.6.2 for the case $\varphi = 0$ Let u be a solution of $P(H, 0)$ and denote by M the graph of u. We know that $u < 0$ on Ω. Since Ω is smooth, Ω satisfies a uniform exterior ϵ-circle condition for a small enough $\epsilon > 0$. Take the family of surfaces $S(c)$ with the same mean curvature H for $r_0 > \epsilon$ and consider the part of $S(c)$ obtained by rotating the function $w(r; c)$ where $r \in (\epsilon, r_0)$. Denote this surface by $S(c)$ again, whose boundary $\partial S(c)$ is formed by two coaxial circles

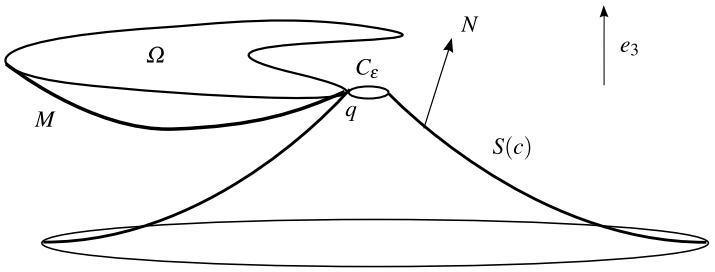

Fig. 12.6 Comparing the H-graph M with the surfaces of type $S(c)$

of radius ϵ and r_0. By the properties of these surfaces, take c sufficiently large and negative such that $r_0 - \epsilon > \delta(\Omega)$. See Fig. 12.6.

Let $q \in \partial\Omega$ and take C_ϵ a circle in $\mathbb{R}^2 \setminus \Omega$ tangent to $\partial\Omega$ at the point q. Translate the surface $S(c)$ horizontally so that its axis of rotation coincides with the vertical line through the center of C_ϵ. Move $S(c)$ down below $\mathbb{R}^2$ until it does not intersect the surface M. Next we move $S(c)$ back by translating vertically upwards. By the height estimates given in (12.24), and the choice of c and ϵ, the tangency principle guarantees that we can position $S(c)$ so that C_ϵ is a boundary component of $\partial S(c)$ and the surface M lies above $S(c)$ in Ω. As $w(q) = u(q) = 0$, it follows that $|\nabla u|(q)$ is bounded by $|\nabla w|(q)$, obtaining the desired a priori gradient estimates of u along $\partial\Omega$. □

Finally, we outline the proof of Theorem 12.6.2 for an arbitrary (spacelike) function φ. Consider the surface $S(c)$ with boundary the circle C_ϵ as in the above proof. We replace the notation w by w^+. Let $w^- = -w^+$, whose graph $S(c)^-$ is the reflection of $S(c)$ through the plane $\mathbb{R}^2$ and $\partial S(c)^- = \partial S(c)^+ = C_\epsilon$. Then $S(c)^-$ has mean curvature $H > 0$ with the downward orientation.

Claim. Let $q \in \partial\Omega$. It is possible to choose $|c|$ sufficiently large and choose ϵ small enough depending only on φ and Ω so that $S(c)^+$ and $S(c)^-$ can be positioned with the following properties:

1. $C_\epsilon \subset \mathbb{R}^2 \setminus \Omega$ with $q \in C_\epsilon$;
2. $w^+ < \varphi - \varphi(q) < w^-$ in $\partial\Omega - \{q\}$.

Having proved the claim, an argument similar to that for the case $\varphi = 0$ proves that we can move $S(c)^+$ and $S(c)^-$ vertically be an amount $\varphi(q)$ until M is positioned between both surfaces and they agree at $(q, \varphi(q))$. Thus

$$|\nabla u|(q) \leq \left|\nabla w^+\right|(q) \leq \max_{\Omega} \left|\nabla w^+\right| = C < 1$$

and $C = C(H, \varphi, \Omega)$.

The proof of the claim is a consequence of a detailed study of the solutions of (12.34): see [BS82, Appendix]. Take the solution of (12.34) with initial conditions $w = w(r_0) = 0$, where $r_0 = \sqrt{-c/H}$. Let $\epsilon = -1/c$ and consider the function

$w(r;\epsilon)$ defined on the interval $(0, r_0)$. Observe that $w'' > 0$ in this interval and that w has a minimum at $r = r_0$. Now (12.34) is

$$w'(r;\epsilon) = \frac{H\epsilon r^2 - 1}{\sqrt{\epsilon^2 r^2 + (Hr^2\epsilon - 1)^2}}.$$

Let $\max_{\overline{\Omega}} |\nabla\varphi| = \mu < 1$ and fix $\mu < \mu_1 < 1$. As $w'(r;\epsilon)$ is strictly increasing on r, consider the value $r_1 = r_1(\epsilon)$ where $w'(r_1;\epsilon) = -\mu_1$. We prove that $\lim_{\epsilon\to 0} r_1(\epsilon) = +\infty$. For small values ϵ,

$$-w'(r) > \frac{1 - H\epsilon r^2}{\sqrt{1 + \epsilon^2 r^2}} > \frac{1 - H\epsilon r^2}{1 + \epsilon r^2}.$$

If we wish $(1 - H\epsilon r^2)/(1 + \epsilon r^2) \geq \mu_1$, then

$$r_1 > \frac{1}{\sqrt{\epsilon}} \sqrt{\frac{1 - \mu_1}{\mu_1 + H}} \to +\infty,$$

as $\epsilon \to 0$.

We prove the claim. Let $q \in \partial\Omega$. Choose ϵ sufficiently small such that $r_1(\epsilon) \gg \delta(\Omega)$. Reducing ϵ if necessary, consider the circle C_ϵ centered at the origin of coordinates and, after a translation, we suppose that $q \in \partial\Omega \cap \partial C_\epsilon$ with $C_\epsilon \subset \mathbb{R}^2 \setminus \Omega$. On Ω we have $|\nabla\varphi| < \mu_1 < |w'| \leq \sup_{\overline{\Omega}} |\nabla w|$. Let $c = -1/\epsilon$. Then $S(c)^+$, with $C_\epsilon \subset \partial S(c)^+$, coincides with φ at q and it lies completely below the graph of φ.

Appendix A
The Variation Formula for the Area and for the Volume

Consider an orientable surface M and let $x : M \to \mathbb{R}^3$ be an isometric immersion. Denote by the same symbol $\langle \cdot, \cdot \rangle$ both the metric of $\mathbb{R}^3$ and the metric induced on M by the immersion x. A *variation of* x is a differentiable map $X : M \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ such that for each $t \in (-\epsilon, \epsilon)$, the maps $x_t : M \to \mathbb{R}^3$ given by $x_t(p) = X(p, t)$ are immersions for all t and for $t = 0$, we have $x_0 = x$. The *variational vector field* $\xi : M \to \mathbb{R}^3$ associated to X is defined by

$$\xi(p) = \frac{\partial X(p,t)}{\partial t}\bigg|_{t=0}(p) = (dX)_{(p,0)}\left(0, \frac{\partial}{\partial t}\right), \quad p \in M.$$

If $U \in \mathfrak{X}(M)$ is a tangent vector field on M, U extends to a vector field along the variation X as

$$\tilde{U}(p,t) = (dX)_{(p,t)}(U_p, 0) = (dx_t)_p(U_p).$$

The covariant derivative of $\tilde{U}$ is defined by

$$\frac{D\tilde{U}}{dt}(p,t) = \frac{D\tilde{U}_p}{dt},$$

where D/dt is the covariant derivative in $\mathbb{R}^3$.

From now, assume that M is a compact surface. Denote the area element of x and x_t by dM and dM_t respectively, and define the functional area $A : (-\epsilon, \epsilon) \to \mathbb{R}^3$ by

$$A(t) = \int_M dM_t.$$

Proposition A.0.1 (First variation for the area) *The function A is differentiable at $t = 0$ with*

$$A'(0) = -2\int_M H\langle N, \xi\rangle \, dM - \int_{\partial M} \langle \nu, \xi\rangle \, ds, \tag{A.1}$$

where ν is the inner conormal vector of M along ∂M and H is the mean curvature of the immersion.

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7,
© Springer-Verlag Berlin Heidelberg 2013

Proof Consider a geodesic frame $\{e_1, e_2\}$ around a point $p \in M$ and let $\{E_1, E_2\}$ be vector fields along the variation X that extend e_1 and e_2 in a neighborhood of p, respectively. Then

$$\begin{aligned} dA_t(p) &= \left(\left|(dx_t)_p(e_1)\right|^2\left|(dx_t)_p(e_2)\right|^2 - \left\langle (dx_t)_p(e_1), (dx_t)_p(e_2)\right\rangle^2\right)^{\frac{1}{2}} \\ &= \sqrt{E(p,t)G(p,t) - F(p,t)^2}, \end{aligned}$$

where we denote by E, F and G the coefficients of the first fundamental form of the immersion x_t. Since M is compact, the parameter dependence theorem gives

$$\begin{aligned} A'(0) &= \int_M \frac{d}{dt}\Big|_{t=0} \sqrt{E(p,t)G(p,t) - F(p,t)^2}\, dM \\ &= \frac{1}{2}\int_M \big(E'(p,0) + G'(p,0)\big)\, dM. \end{aligned}$$

Here we have used that $E(p,0) = |E_1(p,0)|^2 = 1$, $G(p,0) = |E_2(p,0)|^2 = 1$ and $F(p,0) = \langle E_1(p,0), E_2(p,0)\rangle = 0$. Denote by $\tilde{E}_1$ and $\tilde{E}_2$ two vector fields along X extending E_1 and E_2. For the first coefficient E, we find that

$$E(p,t) = \big\langle (dx_t)_p\big(E_1(p,t)\big), (dx_t)_p\big(E_1(p,t)\big)\big\rangle = \big\langle \tilde{E}_1(p,t), \tilde{E}_1(p,t)\big\rangle.$$

Then

$$\frac{d}{dt}\Big|_{t=0} = E'(p,0) = 2\left\langle \frac{D\tilde{E}_1}{dt}(p,0), \tilde{E}_1(p,0)\right\rangle = 2\left\langle \frac{D\tilde{E}_1}{dt}(p,0), (dx)_p(e_1)\right\rangle. \tag{A.2}$$

If $[\cdot,\cdot]$ denotes the Lie bracket in the product manifold $M \times (-\epsilon, \epsilon)$, we see that

$$\left[\big(E_1(p), 0\big), \left(0, \frac{\partial}{\partial t}\right)\right] = 0.$$

Then

$$\left[(dx_t)_p\big(E_1(p), 0\big), (dx_t)_p\left(0, \frac{\partial}{\partial t}\right)\right] = 0,$$

and so,

$$\nabla^0_{\frac{\partial X}{\partial t}(p,t)}(dx_t)_p\big(E_1(p)\big) = \nabla^0_{\tilde{E}_1(p,t)}\frac{\partial X}{\partial t}(p,t),$$

where ∇^0 is the Levi-Civita connection of $\mathbb{R}^3$. At $t = 0$ we obtain

$$\nabla^0_{\xi(p)}\tilde{E}_1 = \nabla^0_{\tilde{E}_1(p,0)}\frac{\partial X}{\partial t}. \tag{A.3}$$

The Gauss formula gives

$$\nabla^0_{e_1}E_1 = \nabla_{e_1}E_1 + \sigma(e_1, e_1)N,$$

where ∇ is the induced connection on M, σ is the second fundamental form of x and N is an orientation on M. We decompose the variational vector field ξ into its tangent and normal parts: $\xi = \xi^\top + \xi^\perp$. Then

$$\begin{aligned}
&\left\langle \nabla_{\tilde{E}_1(p,0)} \frac{\partial X}{\partial t}(p,0), (dx)_p(e_1) \right\rangle \\
&\quad = \left\langle \nabla_{e_1}\xi, (dx)_p(e_1) \right\rangle = e_1\left\langle \xi, dx(E_1) \right\rangle - \left\langle \xi(p), \nabla^0_{e_1} E_1 \right\rangle \\
&\quad = e_1\left\langle \xi^\top, dx(E_1) \right\rangle - \left\langle \nabla_{e_1} E_1, \xi^\top \right\rangle - \sigma(e_1,e_1)\langle N, \xi\rangle \\
&\quad = \langle \nabla_{e_1}\tilde{\xi}, e_1\rangle - \sigma(e_1,e_1)\langle N,\xi\rangle,
\end{aligned}$$

where $\tilde{\xi} \in \mathfrak{X}(M)$ such that $dx_p(\tilde{\xi}_p) = \xi_p^\top$. Finally, (A.2) is now

$$E'(p,0) = 2\langle \nabla_{e_1}\tilde{\xi}, e_1\rangle - 2\sigma(e_1,e_1)\langle N,\xi\rangle.$$

Similarly for the coefficient G, we obtain

$$G'(p,0) = 2\langle \nabla_{e_2}\tilde{\xi}, e_2\rangle - 2\sigma(e_2,e_2)\langle N,\xi\rangle.$$

Then

$$E'(p,0) + G'(p,0) = 2\,\mathrm{div}(\tilde{\xi}) - 4H\langle N,\xi\rangle. \tag{A.4}$$

Using the divergence theorem:

$$A'(0) = \int_M \big(\mathrm{div}(\tilde{\xi}) - 2H\langle N,\xi\rangle\big) dM = -\int_{\partial M} \langle \nu, \tilde{\xi}\rangle ds - 2\int_M H\langle N,\xi\rangle dM,$$

where ν is the unit inward conormal vector of M along the boundary. □

For each t, define the *volume* of x_t as

$$V(t) = -\frac{1}{3}\int_M \left\langle x_t, N(t)\right\rangle dM_t,$$

where $N : M \times (-\epsilon,\epsilon) \to \mathbb{R}^3$ is a differentiable map and $N(t) = N(-,t)$ defines an orientation on x_t.

Proposition A.0.2 (First variation for the volume) *The function V is differentiable at $t = 0$ with*

$$V'(0) = -\int_M \langle N,\xi\rangle dM. \tag{A.5}$$

Proof Using the above notation, the expression of the volume functional is

$$V(t) = -\frac{1}{3}\int_M \langle N(t), x_t\rangle dM_t$$
$$= -\frac{1}{3}\int_M \langle N(t), x_t\rangle \sqrt{E(p,t)G(p,t) - F(p,t)^2}\, dM.$$

The parameter dependence theorem for the function $V(t)$ and (A.4) imply

$$3V'(0) = -\int_M \frac{d}{dt}\Big|_{t=0} \langle N(t), x_t\rangle dM - \int_M (\operatorname{div}\tilde{\xi} - 2Hf)\langle N, x\rangle dM. \tag{A.6}$$

For the first integral,

$$\frac{d}{dt}\Big|_{t=0} \langle N(t), x_t\rangle = \left\langle \frac{d}{dt}\Big|_{t=0} N(t), x \right\rangle + \langle N, \xi\rangle. \tag{A.7}$$

As $\langle N(t), N(t)\rangle = 1$, $\frac{\partial}{\partial t}|_{t=0} N(t)$ is orthogonal to N. On the other hand, if $u \in T_pM$, $\langle N(t)(p), (dx_t)_p(u)\rangle = 0$ and then

$$\left\langle \frac{\partial}{\partial t}\Big|_{t=0} N(t)(p), u \right\rangle + \left\langle N(p), \frac{\partial}{\partial t}\Big|_{t=0} (dx_t)_p(u) \right\rangle = 0. \tag{A.8}$$

Let $U(p,t) = (dx_t)_p(u)$. Then

$$\frac{\partial}{\partial t}\Big|_{t=0} U(p,t) = \frac{D}{\partial t}\Big|_{t=0} U(p,t) = \nabla^0_{E_1(p)} \frac{\partial X}{\partial t} = \nabla^0_u \xi.$$

For each $u \in T_pM$, (A.8) gives

$$\left\langle \frac{\partial}{\partial t}\Big|_{t=0} N(t)(p), u \right\rangle = -\langle N(p), \nabla^0_u \xi\rangle = -u\langle N, \xi\rangle + \langle \nabla_u N, \xi\rangle$$
$$= -u(f) - \langle A(u), \xi\rangle = -u(f) - \langle A(u), \xi^\top\rangle.$$

Since this holds for all $u \in T_pM$, we conclude that

$$\frac{\partial}{\partial t}\Big|_{t=0} N(t)(p) = -\nabla f - A(\xi^\top).$$

Now (A.7) gives

$$\frac{d}{dt}\Big|_{t=0} \langle N(t), x_t\rangle = -\langle \nabla f, x\rangle - \langle A(\xi^\top), x\rangle + f.$$

Thus (A.6) is

$$3V'(0) = -\int_M \left(f - \langle \nabla f, x\rangle - \langle A\xi^\top, x\rangle + \langle N, x\rangle \operatorname{div}\tilde{\xi} - 2Hf\langle N, x\rangle\right) dM. \tag{A.9}$$

On the other hand

$$\begin{aligned}\operatorname{div}\big(\langle N,x\rangle\xi^{\top}\big) &= \langle N,x\rangle \operatorname{div}\xi^{\top} + \big\langle \nabla\langle N,x\rangle, \xi^{\top}\big\rangle = \langle N,x\rangle \operatorname{div}\xi^{\top} + \big\langle dN\big(x^{\top}\big), \xi^{\top}\big\rangle \\ &= \langle N,x\rangle \operatorname{div}\tilde{\xi} - \big\langle Ax^{\top}, \xi^{\top}\big\rangle.\end{aligned}$$

Furthermore

$$\operatorname{div}\big(f\nabla x^{\top}\big) = f\operatorname{div}\big(x^{\top}\big) + \big\langle \nabla f, x^{\top}\big\rangle = f\big(2 + 2H\langle N,x\rangle\big) + \big\langle \nabla f, \nabla x^{\top}\big\rangle.$$

Using the previous two expressions, (A.9) yields

$$\begin{aligned}3V'(0) &= -\int_M \big(3f + \operatorname{div}\big(\langle N,x\rangle\xi^{\top}\big) - \operatorname{div}\big(f\nabla x^{\top}\big)\big)\, dM \\ &= -3\int_M f\, dM + \int_{\partial M}\big(\langle N,x\rangle\langle \nu, \xi^{\top}\rangle + f\langle x^{\top}, \nu\rangle\big)\, ds.\end{aligned}$$

If the variation preserves the boundary, then $\xi = 0$ along ∂M. Thus f and $\xi^{\top}$ vanish along ∂M, obtaining finally $3V'(0) = -\int_M f dM$. □

We now calculate the second derivative for the area $A(t)$. First, we compute $J_H''(0)$.

Proposition A.0.3 (Second variation for the area) *Let M be a compact surface and let $x : M \to \mathbb{R}^3$ be an immersion of constant mean curvature H. Let $X : M \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ be a normal admissible variation of x and $\xi = fN$ the variational vector field. Then*

$$J_H''(0) = -\int_M f\big(\Delta f + |\sigma|^2 f\big)\, dM,$$

where Δf is the Laplacian of f.

Proof For a proof, see also [BC84, Law80, Spi79]. Denote by $H(t)$ the mean curvature of each immersion x_t.

Claim. The derivative of $H(t)$ at $t = 0$ is

$$H'(0) = \frac{1}{2}\big(\Delta f + |\sigma|^2 f\big). \tag{A.10}$$

Assume that the claim is proved. From (A.1), (A.5) and the expression of J_H we have

$$J_H'(t) = -2\int_M \big(H(t) - H\big) f(t)\, dM_t,$$

where

$$f(t) = \left\langle \frac{\partial X}{\partial t}, N(t)\right\rangle.$$

The second derivative of J_H is then

$$J_H''(t) = -2\int_M \frac{\partial(H(t)-H)}{\partial t} f(t)\,dM_t - 2\int_M \big(H(t)-H\big)\frac{\partial(f(t)\,dM_t)}{\partial t}.$$

As at $t=0$, $H(0)=H$, we have

$$J_H''(0) = -2\int_M H'(0) f\,dM$$

and (A.10) proves the result.

Therefore, it remains to prove (A.10). Let $p \in M$ and let $\{e_1, e_2\}$ be a geodesic frame around p. Let $e_i(t)(p) = (dx_t)_p(e_i)$. Then the mean curvature $H(t)$ of x_t is

$$H(t) = \frac{1}{2}\frac{G(t)\langle\nabla^0_{e_1(t)}e_1(t), N(t)\rangle - 2F(t)\langle\nabla^0_{e_1(t)}e_2(t), N(t)\rangle + E(t)\langle\nabla^0_{e_2(t)}e_2(t), N(t)\rangle}{E(t)G(t)-F(t)^2}.$$

We follow the proof of Proposition A.0.1. Since the variation is normal, $\xi^\top = 0$, and thus, $\tilde{\xi} = 0$. As $\langle N, \xi\rangle = f$, we have

$$E'(p,0) = -2\sigma(e_1,e_1)f, \qquad F'(p,0) = -2\sigma(e_1,e_2)f,$$
$$G'(p,0) = -2\sigma(e_2,e_2)f.$$

Since $N'(0)$ is orthogonal to N, $\langle\nabla^0_{e_i}e_i, N'(0)\rangle = 0$. The derivative of $H(t)$ at $t=0$ is:

$$\begin{aligned} H'(0) &= \frac{1}{2}\big(G'(0)\sigma(e_1,e_1) - 2F'(0)\sigma(e_1,e_2) + E'(0)\sigma(e_2,e_2)\big) \\ &\quad - \frac{1}{2}\big(\sigma(e_1,e_1)+\sigma(e_2,e_2)\big)\big(E'(0)+G'(0)\big) \\ &\quad + \frac{1}{2}\left\langle \sum_{i=1}^2 \frac{d}{dt}\bigg|_{t=0} \nabla^0_{e_i(t)}e_i(t), N\right\rangle \\ &= \big(4H^2 - 2K\big)f + \frac{1}{2}\left\langle \sum_{i=1}^2 \frac{d}{dt}\bigg|_{t=0} \nabla^0_{e_i(t)}e_i(t), N\right\rangle. \end{aligned} \tag{A.11}$$

Using that the variation is normal and (A.3), we obtain

$$\begin{aligned} \left\langle \sum_{i=1}^2 \frac{d}{dt}\bigg|_{t=0} \nabla^0_{e_i(t)}e_i(t), N\right\rangle &= \left\langle \sum_{i=1}^2 \nabla^0_\xi \nabla^0_{e_i(t)}e_i(t), N\right\rangle = \left\langle \sum_{i=1}^2 \nabla^0_{e_i}\nabla^0_{e_i(t)}\frac{\partial X}{\partial t}, N\right\rangle \\ &= \left\langle \sum_{i=1}^2 e_i\nabla^0_{e_i(t)}\frac{\partial X}{\partial t}, N\right\rangle - \sum_{i=1}^2 \left\langle \nabla^0_{e_i}\frac{\partial X}{\partial t}, \nabla^0_{e_i}N\right\rangle \end{aligned}$$

$$= \sum_{i=1}^{2} e_i\left(e_i\left(\left\langle \frac{\partial X}{\partial t}, N\right\rangle\right)\right) - f\sum_{i=1}^{2}\langle \nabla^0_{e_i} N, \nabla^0_{e_i} N\rangle$$
$$= \Delta f - f|\sigma|^2.$$

Together with (A.11), this proves the identity (A.10). □

We point out that the statement of Proposition A.0.3 holds for any variation, not necessarily a normal variation: for a proof, see [BC84].

Corollary A.0.4 *Let M be a compact surface and let $x : M \to \mathbb{R}^3$ be an immersion of constant mean curvature H. Let $X : M \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ be a volume preserving normal admissible variation of x and $\xi = fN$ the variational vector field. Then*

$$A''(0) = -\int_M f\left(\Delta + |\sigma|^2 f\right) dM.$$

Proof We know that $J''_H(t) = A''(t) + 2HV''(t)$. As the variation preserves the volume, $V'(t) = 0$, hence, $J''_H(0) = A''(0)$ and we apply Proposition A.0.3. □

Appendix B
Open Questions

We formulate some open problems that arise from the discussions in this book. We exclude those problems that can be proposed in other ambient spaces, arbitrary dimensions or where H is not constant. Here, when we say 'Show' we mean 'Prove or disprove'.

1. Let M be an immersed cmc disk in $\mathbb{R}^3$ spanning a circle. Show that the surface is a planar disk or a spherical cap (Conjecture 1).
2. Let M be a compact embedded cmc surface in $\mathbb{R}^3$ spanning a circle. Show that the surface is a planar disk or a spherical cap (Conjecture 2).
3. Let M be a compact cmc surface in $\mathbb{R}^3$ spanning a circle. Assume that the surface is stable. Show that the surface is a planar disk or a spherical cap (Conjecture 3).
4. By combining two of the three above hypotheses, show that the surface is a planar disk or a spherical cap.
5. The analogous versions of the above conjectures in hyperbolic space $\mathbb{H}^3$. In this case, assume $|H| > 1$. Show that the surface is umbilical.
6. Show that a compact star-shaped cmc surface in $\mathbb{R}^3$ spanning a circle is a spherical cap (boundary version of Jellet theorem).
7. Let Γ_1 and Γ_2 be two coaxial circles in parallel planes and let M be a compact embedded cmc surface in $\mathbb{R}^3$ spanning $\Gamma_1 \cup \Gamma_2$. Show that M is a surface of revolution.
8. Let Γ_1 and Γ_2 be two convex curves in parallel planes and let M be a compact embedded cmc surface in $\mathbb{R}^3$ spanning $\Gamma_1 \cup \Gamma_2$. Show that M is a topological annulus (Meeks conjecture for cmc surfaces).
9. Let Γ be a simple closed convex curve in a plane P. Let M be a compact embedded cmc surface in $\mathbb{R}^3$ with $\partial M = \Gamma$. Assume that M lies on one side of P. Show that M is a topological disk (Conjecture in [BEMR91]).
10. Under the assumptions of the previous item, assume, in addition, that M is a graph on $P \setminus \overline{\Omega}$ around ∂M, where $\Omega \subset P$ is the domain bounded by Γ. Show that M is a bigraph, that is, $M = M_1 \cup M_2$, where M_1 is a graph on a domain $D \subset P$, with $\overline{\Omega} \subset D$ and M_2 is a graph on the annulus $D \setminus \overline{\Omega}$.

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7,
© Springer-Verlag Berlin Heidelberg 2013

11. Let $\Gamma \subset \mathbb{R}^3$ be a simple closed curve included in a plane P. Denote by L its length and $a(\Omega)$ the area of the domain Ω that is bounded by Γ. By the flux formula, we know that $|H| \leq L/(2a(\Omega))$ for any H-surface spanning Γ. Determine whether the value $H = L/(2a(\Omega))$ is always attained, or otherwise find counterexamples.
12. Study the above problem when Γ is an ellipse.
13. Given a bounded domain $\Omega \subset \mathbb{R}^2$, consider H-graphs in $\mathbb{R}^3$ on Ω with boundary $\partial\Omega$. Study when $H_{max} = L(\partial\Omega)/(2a(\Omega))$ (see Corollary 8.1.16).
14. Let M be a compact H-surface immersed in $\mathbb{R}^3$ spanning a circle, $H \neq 0$. If M is included in a slab of width $1/|H|$, show that M is a small spherical cap. If the width of the slab is less than $2/|H|$, show that M is a spherical cap.
15. Let $H > 0$. Show that the only closed H-surface immersed in $\mathbb{R}^3$ included in the closure of a slab of width $2/H$ is a sphere.
16. Consider a compact cmc surface M immersed in $\mathbb{R}^3$ spanning a circle Γ. If the angle between M and the boundary plane is constant along Γ, show that M is a planar disk or a spherical cap.
17. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let M be a compact cmc surface immersed in $\mathbb{R}^3$ spanning $\partial\Omega$. Assume M is included in the solid right cylinder determined by Ω. Show that M is a graph on Ω. Consider stronger assumptions on Ω, for instance that Ω is convex.
18. (Rosenberg conjecture: see [Hof93, p. 21].) Consider a non-circular closed convex curve Γ included in a plane P. Show that there exists a positive number V_Γ depending only on Γ such that, if M is a compact cmc surface in $\mathbb{R}^3$ spanning Γ and with volume $V > 0$, then:

    a. If $V \leq V_\Gamma$, then M lies over the plane P.
    b. If $V = V_\Gamma$, then M is tangent to P at the points of minimum curvature of Γ.
    c. If $V > V_\Gamma$, then M has points on both sides of P.

19. Let Ω be a planar domain. Show that the only compact cmc surface in $\mathbb{R}^3$ with boundary $\partial\Omega$ is a graph if the enclosed volume is sufficiently small.
20. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two bounded convex domains with $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$. Show that there do not exist compact cmc surfaces immersed in $\mathbb{R}^3$ spanning $\partial\Omega_1 \cup \partial\Omega_2$ with volume sufficiently small.
21. Let M be a compact embedded H-surface in $\mathbb{R}^3$ with planar boundary such that its area A satisfies

$$AH^2 < 2\pi.$$

    Show that M is a graph.
22. Let $\Gamma \subset \mathbb{R}^3$ be a simple convex plane curve of length L. Show that there exists a compact H-graph in $\mathbb{R}^3$ spanning Γ if

$$|H| \leq \frac{2\pi}{L}.$$

23. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain of area $a(\Omega)$. Show that there exists a compact H-graph in $\mathbb{R}^3$ spanning $\partial\Omega$ if

$$a(\Omega)H^2 \leq \pi.$$

24. Let $\Omega \subset \mathbb{S}^2$ be a domain. Let $M \subset \mathbb{R}^3$ be a compact radial cmc graph on Ω with boundary $\partial\Omega$. Give conditions to guarantee that M lies on one side of $\mathbb{S}^2$.
25. Let $\Omega \subset \mathbb{R}^2$ be a convex planar domain. Prove that the level curves of a cmc graph in $\mathbb{R}^3$ with boundary $\partial\Omega$ are convex. S. Sakaguchi and J. McCuan gave an affirmative answer assuming, respectively, that the value of the mean curvature H or the enclosed volume by M is sufficiently small [Mcc99, Sak89]. Recently X.-J. Wang has announced a counterexample where the level curves are not all convex [Wng12]. In such a case, give a control of the distribution of the convex level curves. It is known that the height of the graph is less than or equal to $1/|H|$. Show that the levels between the heights $1/(2|H|)$ and $1/|H|$ are convex, as occurs for pieces of unduloids.
26. Let $\Omega \subset \mathbb{R}^2$ be a convex domain and let M be an H-graph on Ω with $\partial M = \partial\Omega$. Give conditions that ensure that the surface is convex. We point out that by cutting off an unduloid with a parallel plane to the axis of rotation in such a way that the compact part contains an inflection point of the generating curve, this surface is a graph with convex plane boundary but there exist points with negative Gaussian curvature K: [Fin88, pp. 189–191]. See also the computer graphics in [Hof93] using Brakke's Surface Evolver program. In fact, the control of the set $M^- = \{p \in M;\, K(p) < 0\}$ allows us to show that if $M^- \neq \emptyset$, then M^- must extend up to the boundary ∂M of the graph [HL98]. In the context of capillarity, see [CY82, Fin83, Kor83a, Kor83b].
27. Extend the monotonicity formula given in Theorem 6.1.1 to hyperbolic space when the boundary lies included in a geodesic plane, an equidistant surface or a horosphere.
28. Let P be a horosphere or an equidistant surface of $\mathbb{H}^3$ and let M be a compact embedded cmc surface in $\mathbb{H}^3$ whose boundary is a convex curve included in P. Show that if M is transverse to P along ∂M, then M lies on one side of P.
29. Let Γ be a simple closed curve in $\mathbb{H}^3$ contained in a horosphere or an equidistant surface. Find an amenable formula for H in terms of Γ for a compact H-surface in $\mathbb{H}^3$ spanning Γ.

References

[AM08] Abraham, R., Marsden, J.: Foundations of Mechanics, 2nd. edn. Am. Math. Soc., Providence (2008) (pp. 3, 83)

[Abr87] Abresch, U.: Constant mean curvature tori in terms of elliptic functions. J. Reine Angew. Math. **394**, 169–192 (1987) (p. 5)

[AR04] Abresch, U., Rosenberg, H.: A Hopf differential for constant mean curvature surfaces in $S^2 \times R$ and $H^2 \times R$. Acta Math. **193**, 141–174 (2004) (p. x)

[AG82] Adamson, A.W., Gast, A.P.: Physical Chemistry of Surfaces, 4th edn. Wiley, New York (1982) (p. 3)

[AF08] Aiolfi, A.J., Fusieger, P.: Some existence results about radial graphs of constant mean curvature with boundary in parallel planes. Ann. Glob. Anal. Geom. **34**, 415–430 (2008) (pp. 33, 165)

[AFR06] Aiolfi, A.J., Fusieger, P., Ripoll, J.: A note on doubly connected surfaces of constant mean curvature with prescribed boundary. Ann. Glob. Anal. Geom. **29**, 145–156 (2006) (pp. 33, 165)

[AM09] Aiolfi, A.J., Mathias, C.V.: Existence and uniqueness of CMC parabolic graphs in $\mathbb{H}^3$ with boundary data satisfying the bounded slope condition. Differ. Geom. Appl. **27**, 755–765 (2009) (pp. 13, 1, 145, 219)

[AR97] Alencar, H., Rosenberg, H.: Some remarks on the existence of hypersurfaces of constant mean curvature with a given boundary, or asymptotic boundary, in hyperbolic space. Bull. Sci. Math. **121**, 61–69 (1997) (pp. 1, 219)

[Ale56] Alexandrov, A.D.: Uniqueness theorems for surfaces in the large I–V. Vestn. Leningr. Univ. **11 #19**, 5–17 (1956); **12 #7**, 15–44, 1957; **13 #7**, 14–26, 1958; **13 #13**, 27–34, 1958; **13 #19**, 5–8, (1958). English transl. in Amer. Math. Soc. Transl. **21**, 341–354, 354–388, 389–403, 403–411, 412–416 (1962) (pp. 4, 20, 1, 32, 55)

[Ale62] Alexandrov, A.D.: A characteristic property of spheres. Ann. Mat. Pura Appl. **58**, 303–315 (1962) (pp. 4, 20, 1, 32, 55)

[Alm66a] Almgren, F.J.: Plateau's Problem: An Invitation to Varifold Geometry, 1st edn. Mathematics Monographs Series. Benjamin, New York (1966) (p. 8)

[Alm66b] Almgren, F.J.: Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. Ann. Math. **84**, 277–292 (1966) (pp. 14, 50)

[Alm76] Almgren, F.J.: Existence and Regularity Almost Everywhere of Solutions to Elliptic Variational Problems with Constraints. Mem. Amer. Math. Soc., vol. 165. Am. Math. Soc., Providence (1976) (pp. 12, 114)

[AD07] Alías, L.J., Dajczer, M.: Normal geodesic graphs of constant mean curvature. J. Differ. Geom. **75**, 387–401 (2007) (pp. 2, 134)

[ADR07] Alías, L.J., Dajczer, M., Rosenberg, H.: The Dirichlet problem for CMC surfaces in Heisenberg space. Calc. Var. Part. Diff. Eq. **30**, 513–522 (2007) (pp. 2, 134)

Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7,

© Springer-Verlag Berlin Heidelberg 2013

[ALP99] Alías, L.J., López, R., Palmer, B.: Stable constant mean curvature surfaces with circular boundary. Proc. Am. Math. Soc. **127**, 1195–1200 (1999) (pp. 1, 123)

[ALT98] Alías, L.J., López, R., Pastor, J.: Constant mean curvature spacelike surfaces in the Lorentz-Minkowski 3-space. Tohoku Math. J. **50**, 491–501 (1998) (pp. 10, 14, 244, 248)

[ALR12] Alías, L.J., López, R., Ripoll, J.: Existence and topological uniqueness of compact cmc hypersurfaces with boundary in hyperbolic space. J. Geom. Anal. (2012). doi:10.1007/s12220-012-9324-2 (pp. 1, 219)

[AM02] Alías, L.J., Malacarne, J.M.: Constant scalar curvature hypersurfaces with spherical boundary in Euclidean space. Rev. Mat. Iberoam. **18**, 431–442 (2002) (pp. 14, 68)

[AP01] Alías, L.J., Palmer, B.: On the area of constant mean curvature discs and annuli with circular boundaries. Math. Z. **237**, 585–599 (2001) (pp. 1, 123)

[AL12] Andrews, B., Li, H.: Embedded constant mean curvature tori in the three-sphere. arXiv:1204.5007v3 (2012) (p. 6)

[Bak94] Bakelman, I.J.: Convex Analysis and Nonlinear Geometric Elliptic Equations. Springer, Berlin (1994) (pp. 1, 133)

[Bar13] Barbosa, A.M.: Compact minimal vertical graphs with non-connected boundary $\mathbb{H}^n \times \mathbb{R}$ (2013). arXiv:1303.3089 (p. x)

[Bar90] Barbosa, J.L.: Hypersurfaces of constant mean curvature on $\mathbb{R}^{n+1}$ bounded by an Euclidean sphere. In: Geometry and Topology II, pp. 1–9. World Sci. Publ., Teaneck (1990) (pp. 12, 92)

[Bar91] Barbosa, J.L.: Constant mean curvature surfaces bounded by a planar curve. Mat. Contemp. **1**, 3–15 (1991) (pp. 14, 94)

[BC78] Barbosa, J.L., do Carmo, M.: A proof of a general isoperimetric inequality for surfaces. Math. Z. **162**, 245–261 (1978) (pp. 6, 1, 108, 123)

[BC80] Barbosa, J.L., do Carmo, M.: Stability of minimal surfaces and eigenvalues of the Laplacian. Math. Z. **173**, 13–28 (1980) (pp. 9, 131)

[BC84] Barbosa, J.L., do Carmo, M.: Stability of hypersurfaces with constant mean curvature. Math. Z. **185**, 339–353 (1984) (pp. 4, 6, 23, 5, 7, 18, 35, 265, 267)

[BaCo86] Barbosa, J.L., Colares, A.G.: Minimal Surfaces in $\mathbb{R}^3$. Lecture Notes in Math., vol. 1195. Springer, Berlin (1986) (pp. x, 3)

[BE98a] Barbosa, J.L., Earp, R.: Prescribed mean curvature hypersurfaces in $\mathbb{H}^{n+1}(-1)$ with convex planar boundary I. Geom. Dedic. **71**, 61–74 (1998) (pp. 23, 1, 211, 219)

[BE98b] Barbosa, J.L., Sa Earp, R.: Prescribed mean curvature hypersurfaces in $\mathbb{H}^{n+1}(-1)$ with convex planar boundary, II. Sém. Théor. Spectr. Géom. **16**, 43–79 (1998) (pp. 23, 211)

[Bar87] Bartnik, R.: Maximal surfaces and general relativity. Miniconference on geometry and partial differential equations, 2. In: Proc. Centre Math. Anal. Austral. Nat. Univ., Canberra, 1986, vol. 12, pp. 24–49. Austral. Nat. Univ., Canberra (1987) (pp. 1, 235)

[BS82] Bartnik, R., Simon, L.: Spacelike hypersurfaces with prescribed boundary values and mean curvature. Commun. Math. Phys. **87**, 131–152 (1982) (pp. 21, 24, 255, 258)

[Bay03] Bayard, P.: Dirichlet problem for space-like hypersurfaces with prescribed scalar curvature in $\mathbb{R}^{n,1}$. Calc. Var. Partial Differ. Equ. **18**, 1–30 (2003) (pp. 21, 255)

[Bgn09] Bergner, M.: On the Dirichlet problem for the prescribed mean curvature equation over general domains. Differ. Geom. Appl. **27**, 335–343 (2009) (pp. 1, 30, 133, 162)

[Brn10] Bernstein, S.N.: Sur la généralization du problème de Dirichlet, II. Math. Ann. **69**, 82–136 (1910) (pp. 11, 91)

[Brn15] Bernstein, S.N.: Sur une théorème de géometrie et ses applications aux équations dérivées partielles du type elliptique. Commun. Soc. Math. Kharkov **15**, 38–45 (1915–1917) (pp. 14, 50)

[Ber51] Bers, L.: Isolated singularities of minimal surfaces. Ann. Math. **53**, 364–386 (1951) (pp. 1, 173)

[BCG02] Bethuel, F., Caldiroli, P., Guida, M.: Parametric surfaces with prescribed mean curvature 2001. Rend. Semin. Mat. (Torino) **60**, 175–231 (2002) (p. 8)
[Bla56] Blaschke, W.: Kreis und Kugel, 2. Auf. Berlin (1956) (pp. 14, 94)
[Bob91] Bobenko, A.I.: All constant mean curvature tori in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$ in terms of theta functions. Math. Ann. **290**, 209–245 (1991) (p. 5)
[BGG] Bombieri, E., De Giorgi, E., Giusti, E.: Minimal cones and the Bernstein problem. Invent. Math. **7**, 243–268 (1969) (pp. 20, 254)
[Boy59] Boys, C.V.: Soap Bubbles: Their Colors and Forces Which Mold Them. Dover, New York (1959) (pp. vii, 3)
[BK12] Breiner, C., Kapouleas, N.: Embedded constant mean curvature surfaces in Euclidean three space (2012). arXiv:1210.3394 (p. 5)
[Bre13] Brendle, S.: Embedded minimal tori in S^3 and the Lawson conjecture. To appear Acta Math. (p. 6)
[BC82] Brézis, H., Coron, J.M.: Sur la conjecture de Rellich pour les surfaces à courbure moyenne prescrite. C. R. Acad. Sci. Paris Sér. I Math. **295**, 615–618 (1982) (p. 9)
[BC84] Brézis, H., Coron, J.M.: Multiple solutions of H-systems and Rellich's conjecture. Commun. Pure Appl. Math. **37**, 149–187 (1984) (p. 9)
[BC85] Brézis, H., Coron, J.M.: Convergence of solutions of H-systems or how to blow bubbles. Arch. Ration. Mech. Anal. **89**, 21–56 (1985) (pp. 19, 121)
[BE91] Brito, F., Earp, R.: Geometric configurations of constant mean curvature surfaces with planar boundary. An. Acad. Bras. Ciênc. **63**, 5–19 (1991) (pp. 9, 89)
[BEMR91] Brito, F., Earp, R., Meeks, W.H. III, Rosenberg, H.: Structure theorems for constant mean curvature surfaces bounded by a planar curve. Indiana Univ. Math. J. **40**, 333–343 (1991) (pp. 17, 20, 9, 97, 100, 269)
[BS80] Brown, R.A., Scriven, L.E.: The shape and stability of rotating liquid drops. Proc. R. Soc. Lond. Ser. A **371**, 331–357 (1980) (pp. 7, 61)
[Bry87] Bryant, R.: Surfaces of mean curvature one in hyperbolic space. Astérisque **154–155**, 321–347 (1987) (pp. 1, 189)
[CF80] Caffarelli, L.A., Friedman, A.: The shape of axisymmetric rotating fluid. J. Funct. Anal. **35**, 109–142 (1980) (pp. 7, 61)
[CG13] Caldiroli, P., Gullino, G.: Radial graphs over domains of $\mathbb{S}^n$ with prescribed mean curvature. J. Fixed Point Theory Appl. (2013). doi:10.1007/s11784-013-0107-6 (pp. 33, 165)
[Car76] do Carmo, M.: Differential Geometry of Curves and Surfaces. Prentice-Hall, Saddle River (1976) (p. viii)
[Car92] do Carmo, M.: Riemannian Geometry. Birkhäuser, Boston (1992) (pp. viii, 15, 27)
[Cha65] Chandrasekhar, S.: The stability of a rotating liquid drop. Proc. R. Soc. Lond. Ser. A **286**, 1–26 (1965) (pp. 7, 61)
[Chv94] Chavel, I.: Riemannian Geometry: A Modern Introduction. Cambridge University Press, New York (1994) (pp. viii, 15, 27)
[CY82] Chen, J.T., Huang, W.H.: Convexity of capillary surfaces in the outer space. Invent. Math. **61**, 253–259 (1982) (pp. 11, 271)
[Che76] Cheng, S.Y.: Eigenfunctions and nodal sets. Comment. Math. Helv. **51**, 43–55 (1976) (pp. 2, 6, 104, 128)
[ChYa76] Cheng, S.Y., Yau, S.T.: Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces. Ann. Math. **104**, 407–419 (1976) (pp. 20, 254)
[Cho02] Choe, J.: Sufficient conditions for constant mean curvature surfaces to be round. Math. Ann. **323**, 143–156 (2002) (pp. 13, 67)
[CY80] Choquet-Bruhat, Y., York, J.: The Cauchy Problem. In: Held, A. (ed.) General Relativity and Gravitation. Plenum, New York (1980) (pp. 1, 235)
[CS12] Citti, G., Senni, C.: Constant mean curvature graphs on exterior domains of the hyperbolic plane. Math. Z. **272**, 531–550 (2012) (p. x)
[CM99] Colding, T., Minicozzi, W.: In: Minimal Surfaces. Courant Lect. Notes Math., vol. 4, New York (1999) (p. 3)

[CM11] Colding, T., Minicozzi, W.: A Course in Minimal Surfaces. Graduate Studies in Mathematics, vol. 121. Am. Math. Soc., Providence (2011) (pp. x, 3)
[Col90] Collin, P.: Deux examples de graphes de courbure moyenne constante sur une bande de $\mathbb{R}^2$. C. R. Math. Acad. Sci. Paris **311**, 539–542 (1990) (pp. 10, 13, 182, 185)
[CK91] Collin, P., Krust, R.: Le problème de Dirichlet pur l'équation des surfaces minimales sur des domains non bornées. Bull. Soc. Math. Fr. **119**, 443–462 (1991) (pp. 1, 10, 14, 173, 182, 186)
[Cou50] Courant, R.: Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces. Interscience, New York (1950) (p. 8)
[CH89] Courant, R., Hilbert, D.: Methods of Mathematical Physics. Wiley, New York (1989) (pp. 10, 11, 6, 8, 9, 3, 7, 22, 23, 128, 130, 131, 135, 179)
[Cus09] Cuschieri, T.: Complete noncompact cmc surfaces in hyperbolic 3-Space. Ph. Doctoral thesis, University of Warwick (2009) (pp. 1, 219)
[Daj06] Dajczer, M.: Constant mean curvature surfaces in warped product spaces. In: XIV Fall Workshop on Geometry and Physics. Publ. R. Soc. Mat. Esp., vol. 10, Madrid, pp. 85–95 (2006) (p. x)
[Daj09] Dajczer, M.: Graphs with prescribed mean curvature. In: Álvarez, J., García-Río, E. (eds.) Differential Geometry. Proc. 8th International Colloquium, pp. 111–119. World Scientific, Hackensack (2009) (pp. 1, 133)
[DHL08] Dajczer, M., Hinojosa, P.A., Lira, J.H.: Killing graphs with prescribed mean curvature. Calc. Var. Partial Differ. Equ. **33**, 231–248 (2008) (pp. 2, 134)
[DL09] Dajczer, M., Lira, J.H.: Killing graphs with prescribed mean curvature and Riemannian submersions. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **26**, 763–775 (2009) (pp. 2, 134)
[DR05] Dajczer, M., Ripoll, J.: An extension of a theorem of Serrin to graphs in warped products. J. Geom. Anal. **15**, 193–205 (2005) (pp. 2, 1, 134, 219)
[DR11] Dajczer, M., Ripoll, J.: Constant mean curvature hypersurfaces with single valued projections on planar domains. J. Differ. Equ. **250**, 1493–1499 (2011) (pp. 1, 133)
[DHM09] Daniel, B., Hauswirth, L., Mira, P.: Constant mean curvature surfaces in homogeneous 3-manifolds. In: Lectures Notes of the 4th KIAS Workshop on Differential Geometry "Constant Mean Curvature Surfaces in Homogeneous Manifolds", Seoul (2009) (p. x)
[Del41] Delaunay, C.: Sur la surface de révolution dont la courbure moyenne est constante. J. Math. Pures Appl. **6**, 309–315 (1841) (pp. 5, 9, 11, 25, 6, 65, 157, 178)
[DHKW92] Dierkes, U., Hildebrandt, S., Küster, A., Wohlrab, O.: Minimal Surfaces I, II. Springer, Berlin (1992) (pp. x, 3, 8, 1, 133)
[DHS10] Dierkes, U., Hildebrandt, S., Sauvigny, F.: Minimal Surfaces, Grundlehren der mathematischen Wissenschaften, vol. 339. Springer, Berlin (2010) (pp. 1, 133)
[DH98] Dorfmeister, J., Haak, G.: On constant mean curvature surfaces with periodic metric. Pac. J. Math. **182**, 229–287 (1998) (p. 5)
[DH00] Dorfmeister, J., Haak, G.: Investigation and application of the dressing action on surfaces of constant mean curvature. Q. J. Math. **51**, 57–73 (2000) (p. 5)
[DH03] Dorfmeister, J., Haak, G.: Construction of non-simply-connected CMC surfaces via dressing. J. Math. Soc. Jpn. **55**, 335–364 (2003) (p. 5)
[DPW98] Dorfmeister, J., Pedit, F., Wu, H.: Weierstrass type representation of harmonic maps into symmetric spaces. Commun. Anal. Geom. **6**, 633–668 (1998) (pp. x, 5)
[DW93] Dorfmeister, J., Wu, H.: Constant mean curvature surfaces and loop groups. J. Reine Angew. Math. **440**, 1–47 (1993) (p. 5)
[DW08] Dorfmeister, J., Wu, H.: Construction of constant mean curvature trinoids from holomorphic potentials. Math. Z. **258**, 773–803 (2008) (p. 5)
[Dou30] Douglas, J.: The mapping theorem of Koebe and the problem of Plateau. J. Math. Phys. **10**, 106–130 (1930–1931) (p. 8)
[Dou31] Douglas, J.: Solution of the problem of Plateau. Trans. Am. Math. Soc. **33**, 263–321 (1931) (p. 8)

[Dou39a] Douglas, J.: The higher topological form of the problem of Plateau. Ann. Sc. Norm. Super. Pisa **8**, 195–218 (1939) (p. 8)
[Dou39b] Douglas, J.: The most general form of the problem of Plateau. Am. J. Math. **61**, 590–608 (1939) (p. 8)
[DuHi94] Duc, D.M., Hieu, N.V.: Graphs with prescribed mean curvature on hyperbolic spaces. Manuscr. Math. **83**, 111–121 (1994) (pp. 2, 220)
[DuHi95] Duc, D.M., Hieu, N.V.: Graphs with prescribed mean curvature on Poincaré disk. Bull. Lond. Math. Soc. **27**, 353–358 (1995) (pp. 2, 220)
[DS94] Duc, D.M., Salavesa, I.M.C.: On a class of graphs with prescribed mean curvature. Manuscr. Math. **82**, 227–239 (1994) (pp. 2, 220)
[DuSt96] Duzaar, F., Steffen, K.: The Plateau problem for parametric surfaces with prescribed mean curvature. In: Jost, J. (ed.) Geometric Analysis and the Calculus of Variations, pp. 13–70. International Press, Cambridge (1996) (p. 8)
[EN96] Earp, R., Nelli, B.: Some properties of hypersurfaces of prescribed mean curvature in $\mathbb{H}^{n+1}$. Bull. Sci. Math. **120**, 537–553 (1996) (pp. 2, 220)
[ER89] Earp, R., Rosenberg, H.: The Dirichlet problem for the minimal surface equation on unbounded planar domains. J. Math. Pures Appl. **68**, 163–183 (1989) (pp. 1, 11, 173, 183)
[ER91] Earp, R., Rosenberg, H.: Some structure theorems for complete constant mean curvature surfaces with boundary a convex curve. Proc. Am. Math. Soc. **113**, 1045–1053 (1991) (pp. 6, 60)
[ET99] Earp, R., Toubiana, E.: Some applications of maximum principle to hypersurfaces in Euclidean and hyperbolic space. In: Rassias, T.M. (ed.) New Approaches in Nonlinear Analysis, pp. 183–202. Hardonic Press (1999) (pp. 1, 219)
[ET00] Earp, R., Toubiana, E.: Minimal graphs in hyperbolic space. Asian J. Math. **4**, 669–694 (2000) (pp. 1, 219)
[ET01] Earp, R., Toubiana, E.: Variants on Alexandrov reflection principle and other applications of maximum principle. Sém. Théor. Spectr. Géom. **19**, 93–121 (2001) (pp. 2, 220)
[ET08] Earp, R., Toubiana, E.: An asymptotic theorem for minimal surfaces and existence results for minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$. Math. Ann. **342**, 309–331 (2008) (p. x)
[ET11] Earp, R., Toubiana, E.: Minimal graphs in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{R}^{n+1}$. Ann. Inst. Fourier (Grenoble) **60**(7), 2373–2402 (2010) (pp. x, 2, 134)
[EJ07] Eschenburg, J., Jost, J.: Differentialgeometrie und Minimalflächen. Springer, Berlin (2007) (p. vii)
[EFR10] Espirito-Santo, N., Fornari, S., Ripoll, J.: The Dirichlet problem for the minimal hypersurface equation in $M \times R$ with prescribed asymptotic boundary. J. Math. Pures Appl. **93**, 204–221 (2010) (pp. x, 2, 134)
[EsRi01] Espirito-Santo, N., Ripoll, J.: Some existence and nonexistence theorems for compact graphs of constant mean curvature with boundary in parallel planes. J. Geom. Anal. **11**, 603–618 (2001) (pp. 1, 133)
[EsRi11] Espirito-Santo, N., Ripoll, J.: Some existence results on the exterior Dirichlet problem for the minimal hypersurface equation. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **28**, 385–393 (2011) (pp. 1, 173)
[Fed69] Federer, H.: Geometric Measure Theory. Springer, New York (1969) (pp. 8, 2, 20, 104)
[Fin54] Finn, R.: On equations of minimal surface type. Ann. Math. **60**, 397–416 (1954) (pp. 11, 10, 142)
[Fin65] Finn, R.: Remarks relevant to minimal surfaces and to surfaces of prescribed mean curvature. J. Anal. Math. **14**, 139–160 (1965) (pp. 11, 14, 10, 2, 50, 91, 142, 174, 182)
[Fin83] Finn, R.: Existence criteria for capillary free surfaces without gravity. Indiana Univ. Math. J. **32**, 439–460 (1983) (pp. 11, 271)

[Fin84] Finn, R.: A limiting geometry for capillary surfaces. Ann. Sc. Norm. Super. Pisa **11**, 3671–3790 (1984) (pp. 27, 159)
[Fin86] Finn, R.: Equilibrium Capillary Surfaces. Springer, Berlin (1986) (pp. x, 3, 12, 9, 13, 63, 67)
[Fin88] Finn, R.: Comparison principles in capillarity. In: Partial Differential Equations and Calculus of Variations. Lectures Notes in Mathematics, vol. 1357, pp. 156–197. Springer, Berlin (1988) (pp. 27, 11, 159, 271)
[FS80] Fischer-Colbrie, D., Schoen, R.: The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. Commun. Pure Appl. Math. **33**, 199–211 (1980) (pp. 22, 4, 34, 176)
[FM11] Folha, A., Melo, S.: The Dirichlet problem for constant mean curvature graphs in $\mathbb{H} \times \mathbb{R}$ over unbounded domains. Pac. J. Math. **251**, 37–65 (2011) (pp. x, 2, 1, 134, 173)
[FR12] Folha, A., Rosenberg, H.: The Dirichlet problem for constant mean curvature graphs in $M \times R$. Geom. Topol. **16**, 1171–1203 (2012) (pp. x, 2, 134)
[FT91] Fomenko, A., Tuzhilin, A.: Elements of the Geometry and Topology of Minimal Surfaces in Three-Dimensional Space. Translations of Mathematical Monographs, vol. 93. Am. Math. Soc., Providence (1991) (p. 3)
[FLR02] Fornari, S., Lira, J.H., Ripoll, J.: Geodesic graphs with constant mean curvature in spheres. Geom. Dedic. **90**, 201–216 (2002) (pp. 1, 133)
[FKR05] Fujimori, S., Kobayashi, S., Rossman, W.: In: Loop Groups Methods for Constant Mean Curvature Surfaces. Rokko Lectures in Mathematics, vol. 17 (2005). Kobe (pp. x, 5)
[FR03] Fusieger, P., Ripoll, J.: Radial graphs of constant mean curvature and doubly connected minimal surfaces with prescribed boundary. Ann. Glob. Anal. Geom. **23**, 373–400 (2003) (pp. 13, 33, 145, 165)
[GM00a] Gálvez, J.A., Martínez, A.: The Gauss map and second fundamental form of surfaces in R^3. Geom. Dedic. **81**, 181–192 (2000) (pp. 19, 31)
[GM00b] Gálvez, J.A., Martínez, A.: Estimates in surfaces with positive constant Gauss curvature. Proc. Am. Math. Soc. **128**, 3655–3660 (2000) (pp. 19, 31)
[GBQ04] de Gennes, P., Brochard-Wyart, F., Quéré, D.: Capillarity and Wetting Phenomena. Springer, New York (2004) (p. 3)
[Ger31] Germain, S.: Mémoire sur la courbure des surfaces. J. Reine Angew. Math. **7**, 1–29 (1831) (ed. A.L. Crelle) (p. 3)
[GNN79] Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry and related properties via the maximum principle. Commun. Math. Phys. **68**, 209–243 (1979) (pp. 13, 67)
[GT01] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001). Reprint of the 1998 edition (pp. x, 17, 8, 14, 1, 2, 5, 6, 10, 7, 9, 29, 44, 116, 133, 134, 137, 138, 142, 179, 181)
[GH81] Greenberg, M., Harper, J.: Algebraic Topology: A First Course. Benjamin-Cummings, Reading (1981) (pp. 6, 16, 42, 70)
[Gri00] Grigor'eva, E.G.: On the existence of space-like surfaces with a given boundary. Sib. Mat. Zh. **41**, 1039–1045 (2000) (Russian), translation in Siberian Math. J. **41**, 849–854 (2000) (pp. 21, 255)
[Gro96] Grosse-Brauckmann, K.: Stable constant mean curvature surfaces minimize area. Pac. J. Math. **175**, 527–534 (1996) (pp. 23, 35)
[GK99] Grosse-Brauckmann, K., Kusner, R.: Embedded constant mean curvature surfaces with special symmetry. Manuscr. Math. **99**, 135–150 (1999) (p. 5)
[GKS00] Grosse-Brauckmann, K., Kusner, R., Sullivan, J.: Constant mean curvature surfaces with three ends. Proc. Natl. Acad. Sci. USA **97**, 14067–14068 (2000) (p. 5)
[GKS03] Grosse-Brauckmann, K., Kusner, R., Sullivan, J.: Triunduloids: embedded constant mean curvature surfaces with three ends and genus zero. J. Reine Angew. Math. **564**, 35–61 (2003) (p. 5)

[GKS07] Grosse-Brauckmann, K., Kusner, R., Sullivan, J.: Coplanar constant mean curvature surfaces. Commun. Anal. Geom. **15**, 985–1023 (2007) (p. 5)
[GS00] Guan, B., Spruck, J.: Hypersurfaces of constant mean curvature in hyperbolic space with prescribed asymptotic boundary at infinity. Am. J. Math. **122**, 1039–1060 (2000) (pp. 2, 220)
[GE05] Guio, E., Earp, R.: Existence and non-existence for a mean curvature equation in hyperbolic space. Commun. Pure Appl. Anal. **4**, 549–568 (2005) [Erratum: Commun. Pure Appl., Anal. **7**, 465 (2008)] (pp. 1, 219)
[GuSp71] Gulliver, R., Spruck, J.: The Plateau problem for surfaces of prescribed mean curvature in a cylinder. Invent. Math. **13**, 169–178 (1971) (p. 8)
[GuSp72] Gulliver, R., Spruck, J.: Existence theorems for parametric surfaces of prescribed mean curvature. Indiana Univ. Math. J. **22**, 445–472 (1972) (p. 8)
[Har66] Hartman, P.: On the bounded slope condition. Pac. J. Math. **18**, 495–511 (1966) (pp. 11, 143)
[HRS09] Hauswirth, L., Rosenberg, H., Spruck, J.: Infinite boundary value problems for constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. Am. J. Math. **131**, 195–226 (2009) (pp. x, 2, 134)
[Hea63] Heath, T.L.: Greek Mathematics. Dover, New York (1963) (pp. 5, 107)
[Hei54] Heinz, E.: Über die Existenz einer Fläche konstanter mittlerer Krümmung belvorgegebener Berandung. Math. Ann. **127**, 258–287 (1954) (p. 8)
[Hei69] Heinz, E.: On the nonexistence of a surface of constant mean curvature with finite area and prescribed rectifiable boundary. Arch. Ration. Mech. Anal. **35**, 249–252 (1969) (pp. 9, 7, 87)
[Hel01] Hélein, F.: Constant Mean Curvature Surfaces, Harmonic Maps and Integrable Systems. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel (2001) (pp. x, 5)
[HC99] Hilbert, D., Cohn-Vossen, S.: Geometry and the Imagination. AMS Chelsea Publ., Providence (1999) (pp. 9, 10)
[Hil70] Hildebrandt, S.: On the Plateau problem for surfaces of constant mean curvature. Commun. Pure Appl. Math. **23**, 97–114 (1970) (pp. 8, 12, 92)
[HK72] Hildebrandt, S., Kaul, H.: Two-dimensional variational problems with obstructions, and Plateau's problem for H-surfaces in a Riemannian manifold. Commun. Pure Appl. Math. **25**, 187–223 (1972) (p. 8)
[Hin02] Hinojosa, P.: Graphs with constant mean curvature in the 3-hyperbolic space. An. Acad. Bras. Ciênc. **74**, 371–377 (2002) (pp. 22, 210)
[Hit00] Hitchin, N.J.: Global differential geometry. In: Enquist, B., Schmid, W. (eds.) Mathematics Unlimited—2001 and Beyond, pp. 577–591. Springer, Heidelberg (2000) (p. 2)
[Hof93] Hoffman, D.: Comment utiliser un ordinateur pour trouver de nouvelles surfaces minimales et des bulles de savon. Surfaces minimales et solutions de problèmes variationnels. Soc. Math. France, Paris (1993). 29 pp. (pp. 10, 11, 270, 271)
[Hof05] Hoffman, D.: Global Theory of Minimal Surfaces. Clay Mathematics Proceedings, vol. 2. Am. Math. Soc., Providence (2005) (p. 3)
[HO82] Hoffman, D., Osserman, R.: The area of the generalized Gaussian image and the stability of minimal surfaces in S^n and R^n. Math. Ann. **260**, 437–452 (1982) (pp. 9, 131)
[HRS92] Hoffman, D., Rosenberg, H., Spruck, J.: Boundary value problems for surfaces of constant Gauss curvature. Commun. Pure Appl. Math. **45**, 1051–1062 (1992) (pp. 8, 62)
[Hop27] Hopf, E.: Elementare Bemerkumgen Über Die Lösungen Partieller Differential Gleichungen. S. B. Preuss. Akad. Phys. Math. Kl., 147–152 (1927) (pp. 9, 45)
[Hop52] Hopf, E.: Über Die Lösungen del Minimalflächengleichung. Nachr. Akad. Wiss. Gött. Math.-Phys. Kl., 2B **II**, 51–56 (1952) (pp. 14, 50)
[Hop83] Hopf, H.: Differential Geometry in the Large. Lecture Notes in Mathematics, vol. 1000. Springer, Berlin (1983) (pp. vii, 4, 14, 20, 12, 1, 6, 26, 32, 48, 55, 60,

68)
[HL98] Huang, W.H., Lin, C.C.: Negatively curved sets in surfaces of constant mean curvature in $\mathbb{R}^3$. Arch. Ration. Mech. Anal. **141**, 105–116 (1998) (pp. 11, 271)
[Hwa88] Hwang, J.F.: Comparisons principles and theorems for prescribed mean curvature equation in unbounded domains. Ann. Sc. Norm. Super. Pisa **15**, 341–355 (1988) (pp. 1, 14, 173, 186)
[Hwa94] Hwang, J.F.: Growth property for the minimal surface equation in unbounded domains. Proc. Am. Math. Soc. **121**, 1027–1037 (1994) (pp. 1, 173)
[Ise92] Isenberg, C.: The Science of Soap Films and Soap Bubbles. Dover, New York (1992) (p. 3)
[Jel53] Jellet, J.H.: Sur la surface dont la courbure moyenne est constant. J. Math. Pures Appl. **18**, 163–167 (1853) (pp. 4, 19, 31)
[JS63] Jenkins, H., Serrin, J.: Variational problems of minimal surface type, II. Boundary value problems for the minimal surface equation. Arch. Ration. Mech. Anal. **12**, 185–212 (1963) (pp. 11, 183)
[Jin06] Jin, Z.: Existence of solutions of the prescribed mean curvature equation on unbounded domains. Proc. R. Soc. Edinb., Sect. A, Math. **136**, 157–179 (2006) (pp. 1, 173)
[Jin08] Jin, Z.: Growth rate and existence of solutions to Dirichlet problems for prescribed mean curvature equations on unbounded domains. Electron. J. Differ. Equ. **24**, 15 (2008) (pp. 1, 173)
[Jle09] Jleli, M.: End-to-end gluing of constant mean curvature hypersurfaces. Ann. Fac. Sci. Toulouse **18**, 717–737 (2009) (p. 5)
[JP05] Jleli, M., Pacard, F.: An end-to-end construction for compact constant mean curvature surfaces. Pac. J. Math. **221**, 81–108 (2005) (p. 5)
[Kap90] Kapouleas, N.: Complete constant mean curvature surfaces in Euclidean three-space. Ann. Math. **131**, 239–330 (1990) (pp. x, 5)
[Kap91] Kapouleas, N.: Compact constant mean curvature surfaces in Euclidean three-space. J. Differ. Geom. **33**, 683–715 (1991) (pp. x, 5, 6, 10, 1, 17, 55, 71)
[Kap92] Kapouleas, N.: Constant mean curvature surfaces constructed by fusing Wente tori. Proc. Natl. Acad. Sci. USA **89**, 5695–5698 (1992) (pp. x, 5)
[Kap11] Kapouleas, N.: Doubling and desingularization constructions for minimal surfaces. In: Surveys in Geometric Analysis and Relativity. Adv. Lect. Math., vol. 20, pp. 281–325. International Press, Somerville (2011) (p. 6)
[KY10] Kapouleas, N., Yang, S.D.: Minimal surfaces in the three-sphere by doubling the Clifford torus. Am. J. Math. **132**, 257–295 (2010) (p. 6)
[KPS88] Karcher, H., Pinkall, U., Sterling, I.: New minimal surfaces in S^3. J. Differ. Geom. **28**, 169–185 (1988) (p. 6)
[Ken03] Kenmotsu, K.: Surfaces with Constant Mean Curvature. Translations of Mathematical Monographs, vol. 221. Am. Math. Soc., Providence (2003) (p. vii)
[KKRS05] Kilian, M., Kobayashi, S.P., Rossman, W., Schmitt, N.: Constant mean curvature surfaces of any positive genus. J. Lond. Math. Soc. **72**, 258–272 (2005) (p. 5)
[KMS00] Kilian, M., McIntosh, I., Schmitt, N.: New constant mean curvature surfaces. Expo. Math. **9**, 595–611 (2000) (p. 5)
[KSS04] Kilian, M., Schmitt, N., Sterling, I.: Dressing CMC n-noids. Math. Z. **246**, 501–519 (2004) (p. 5)
[Kly95] Klyachin, A.A.: Solvability of the Dirichlet problem for the equation of maximal surfaces with singularities in unbounded domains. Dokl. Akad. Nauk SSSR **342**, 162–164 (1995) (Russian) (pp. 21, 255)
[Koi86] Koiso, M.: Symmetry of hypersurfaces of constant mean curvature with symmetric boundary. Math. Z. **191**, 567–574 (1986) (pp. 6, 2, 42, 56, 60)
[Koi95] Koiso, M.: A generalization of Steiner symmetrization for immersed surfaces and its applications. Manuscr. Math. **87**, 311–325 (1995) (pp. 13, 115)

[Koi02] Koiso, M.: Deformation and stability of surfaces with constant mean curvature. Tohoku Math. J. **54**, 145–159 (2002) (pp. 23, 5, 35, 137)
[KP13] Koiso, M., Palmer, B.: Stable surfaces with constant anisotropic mean curvature and circular boundary. Proc. Amer. Math. Soc., to appear (pp. 8, 130)
[Kor83a] Korevaar, N.J.: Capillary surface convexity above convex domains. Indiana Univ. Math. J. **32**, 73–81 (1983) (pp. 11, 271)
[Kor83b] Korevaar, N.J.: Convex solutions to nonlinear elliptic and parabolic boundary value problems. Indiana Univ. Math. J. **32**, 603–614 (1983) (pp. 11, 271)
[KK93] Korevaar, N., Kusner, R.: The global structure of constant mean curvature surfaces. Invent. Math. **114**, 311–332 (1993) (p. 5)
[KKMS92] Korevaar, N., Kusner, R., Meeks, W. III, Solomon, B.: Constant mean curvature surfaces in hyperbolic space. Am. J. Math. **114**, 1–43 (1992) (pp. 16, 10, 12, 28, 198, 230)
[KKR06] Korevaar, N., Kusner, R., Ratzkin, J.: On the nondegeneracy of constant mean curvature surfaces. Geom. Funct. Anal. **16**, 891–923 (2006) (p. 5)
[KKS89] Korevaar, N., Kusner, R., Solomon, B.: The structure of complete embedded surfaces with constant mean curvature. J. Differ. Geom. **30**, 465–503 (1989) (p. 5)
[Kou72] Koutroufiotis, D.: On Blaschke's rolling theorems. Arch. Math. **23**, 655–660 (1972) (pp. 14, 94)
[Kou73] Koutroufiotis, D.: Elementary geometric applications of a maximum principle for nonlinear elliptic operators. Arch. Math. **24**, 97–99 (1973) (pp. 17, 53)
[Kus88] Kusner, R.: Global geometry of extremal surfaces in three-space. Ph. Doctoral thesis, California University, Berkeley (1988) (pp. 1, 4, 81, 84)
[Kus91] Kusner, R.: Bubbles, conservation laws, and balanced diagrams. In: Geometric Analysis and Computer Graphics, Berkeley, CA, 1988. Math. Sci. Res. Inst. Publ., vol. 17, pp. 103–108. Springer, New York (1991) (pp. 1, 81)
[KMP96] Kusner, R., Mazzeo, R., Pollack, D.: The moduli space of complete embedded constant mean curvature surfaces. Geom. Funct. Anal. **6**, 120–137 (1996) (p. 5)
[KT98] Kutev, N., Tomi, F.: Existence and nonexistence for the exterior Dirichlet problem for the minimal surface equation in the plane. Differ. Integral Equ. **11**, 917–928 (1998) (pp. 1, 173)
[Kuw93] Kuwert, E.: On solutions of the exterior Dirichlet problem for the minimal surface equation. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **10**, 445–451 (1993) (pp. 1, 173)
[Lan02] Langbein, D.: Capillary Surfaces: Shape—Stability—Dynamics in Particular Under Weightlessness. Springer, Berlin (2002) (pp. x, 3, 12, 7, 61)
[LR88] Langevin, R., Rosenberg, H.: A maximum principle at infinity for minimal surfaces and applications. Duke Math. J. **57**, 819–826 (1988) (pp. 1, 173)
[Lap05] Laplace, P.S.: Traité de mécanique céleste; suppléments au Livre X. Oeuvres Complete, vol. 4. Gauthier-Villars, Paris (1805) (p. 3)
[Law70a] Lawson, H.B.: Complete minimal surfaces in S^3. Ann. Math. **92**, 335–374 (1970) (p. 6)
[Law70b] Lawson, H.B.: The unknottedness of minimal embeddings. Invent. Math. **11**, 183–187 (1970) (p. 6)
[Law80] Lawson, H.B.: Lectures on Minimal Submanifolds, vol. I. Publish or Perish, Berkeley (1980) (pp. vii, 3, 5, 265)
[Lie00] Liebmann, H.: Über die Verbiebung der geschlossenen Flächen positiver Krümmung. Math. Ann. **53**, 91–112 (1900) (pp. 4, 2, 220)
[Lir02a] Lira, J.H.S.: Embedded hypersurfaces in $\mathbb{S}^{n+1}$ with constant mean curvature and spherical boundary. Ann. Glob. Anal. Geom. **21**, 123–133 (2002) (pp. 17, 205)
[Lir02b] Lira, J.H.S.: Radial graphs with constant mean curvature in the hyperbolic space. Geom. Dedic. **93**, 11–23 (2002) (pp. 1, 219)
[LLS00] López, F.J., López, R., Souam, R.: Maximal surfaces of Riemann type in Lorentz–Minkowski space $\mathbb{L}^3$. Mich. Math. J. **47**, 469–497 (2000) (pp. 9, 243)

[Lop96] López, R.: Superficies con Curvatura Media Constante en el Espacio Euclídeo. Tesis Doctoral, Universidad de Granada, Granada (1996) (p. xi)
[Lop97a] López, R.: Surfaces of constant mean curvature bounded by two planar curves. Ann. Glob. Anal. Geom. **15**, 201–210 (1997) (pp. 10, 18, 64, 72)
[Lop97b] López, R.: A note on H-surfaces with boundary. J. Geom. **60**, 80–84 (1997) (pp. 14, 68)
[Lop97c] López, R.: Surfaces of constant mean curvature bounded by convex curves. Geom. Dedic. **66**, 255–263 (1997) (pp. 15, 95)
[Lop97d] López, R.: Surfaces of constant mean curvature with boundary in a sphere. Osaka Math. J. **34**, 573–577 (1997) (pp. 23, 77)
[Lop99a] López, R.: Constant mean curvature surfaces bounded by a circle. Rocky Mt. J. Math. **29**, 971–978 (1999) (pp. 10, 132)
[Lop99b] López, R.: Constant mean curvature surfaces with boundary in Euclidean three-space. Tsukuba J. Math. **23**, 27–36 (1999) (pp. 8, 1, 13, 88, 133, 145)
[Lop99c] López, R.: On uniqueness of constant mean curvature surfaces with planar boundary. In: New Developments in Differential Geometry, Budapest, 1996, pp. 235–242. Kluwer Academic, Dordrecht (1999) (pp. 9, 141)
[Lop99d] López, R.: Constant mean curvature surfaces with boundary in hyperbolic space. Monatshefte Math. **127**, 155–169 (1999) (pp. 2, 23, 190, 211)
[Lop01a] López, R.: Graphs of constant mean curvature in hyperbolic space. Ann. Glob. Anal. Geom. **20**, 59–75 (2001) (pp. 1, 219)
[Lop01b] López, R.: Constant mean curvature graphs on unbounded convex domains. J. Differ. Equ. **171**, 54–62 (2001) (pp. 1, 13, 133, 145, 173)
[Lop02a] López, R.: An existence theorem of constant mean curvature graphs in Euclidean space. Glasg. Math. J. **44**, 455–461 (2002) (pp. 1, 13, 133, 145)
[Lop02b] López, R.: Constant mean curvature graphs in a strip of $\mathbb{R}^2$. Pac. J. Math. **206**, 359–374 (2002) (pp. 1, 173)
[Lop03a] López, R.: A note on radial graphs with constant mean curvature. Manuscr. Math. **110**, 45–54 (2003) (pp. 34, 166)
[Lop03b] López, R.: Some a priori bounds for solutions of the constant Gauss curvature equation. J. Differ. Equ. **194**, 185–197 (2003) (pp. 19, 31)
[Lop04] López, R.: Area monotonicity for spacelike surfaces with constant mean curvature. J. Geom. Phys. **52**, 353–363 (2004) (pp. 17, 251)
[Lop05] López, R.: Wetting phenomena and constant mean curvature surfaces with boundary. Rev. Math. Phys. **17**, 769–792 (2005) (p. xi)
[Lop06a] López, R.: Superficies con curvatura media constante cuyo borde es un círculo. Divulg. Mat. **14**, 121–140 (2006) (p. xi)
[Lop06b] López, R.: Symmetry of stationary hypersurfaces in hyperbolic space. Geom. Dedic. **119**, 35–47 (2006) (pp. 15, 203)
[Lop07] López, R.: On uniqueness of graphs with constant mean curvature. J. Math. Kyoto Univ. **46**, 771–787 (2007) (pp. 17, 29)
[Lop09] López, R.: A new proof of a characterization of small spherical caps. Results Math. **55**, 427–436 (2009) (pp. 20, 32)
[Lop10a] López, R.: Surfaces with constant mean curvature in Euclidean space. Int. Electron. J. Geom. **3**, 67–101 (2010) (p. xi)
[Lop10b] López, R.: Stationary rotating surfaces in Euclidean space. Calc. Var. Partial Differ. Equ. **39**, 333–359 (2010) (pp. 7, 61)
[Lop12] López, R.: Constant mean curvature surfaces in Sol with non-empty boundary. Houst. J. Math. **38**, 1091–1105 (2012) (p. x)
[LR89] López, F.J., Ros, A.: Complete minimal surfaces with index one and stable constant mean curvature surfaces. Comment. Math. Helv. **64**, 34–43 (1989) (pp. 4, 176)
[LM95] López, R., Montiel, S.: Constant mean curvature discs with bounded area. Proc. Am. Math. Soc. **123**, 1555–1558 (1995) (pp. 2, 9, 1, 82, 89, 123)

[LM96] López, R., Montiel, S.: Constant mean curvature surfaces with planar boundary. Duke Math. J. **85**, 583–604 (1996) (pp. 2, 1, 13, 82, 103, 123, 133, 145)
[LM99] López, R., Montiel, S.: Existence of constant mean curvature graphs in hyperbolic space. Calc. Var. Partial Differ. Equ. **8**, 177–190 (1999) (pp. 1, 219)
[LP13] López, R., Pyo, J.: Constant mean curvature surfaces with boundary on a sphere. Appl. Math. Comp. **220**, 316–323 (2013) (pp. 18, 72)
[MN13] Marques, F.C., Neves, A.: Min-max theory and the Willmore conjecture. Ann. Math. (2), to appear (p. 6)
[MT80] Marsden, J.E., Tipler, F.J.: Maximal hypersurfaces and foliations of constant mean curvature in general relativity. Phys. Rep. **66**, 109–139 (1980) (pp. 1, 21, 235, 255)
[Maz04a] Mazet, L.: Quelques résultats de non-existence pour l'équation des surfaces minimales. Bull. Sci. Math. **128**, 577–586 (2004) (pp. 1, 173)
[Maz04b] Mazet, L.: The Dirichlet problem for the minimal surfaces equation and the Plateau problem at infinity. J. Inst. Math. Jussieu **3**, 397–420 (2004) (pp. 1, 173)
[Maz07a] Mazet, L.: Uniqueness results for constant mean curvature graphs. Pac. J. Math. **230**, 365–380 (2007) (pp. 14, 186)
[Maz07b] Mazet, L.: A uniqueness result for maximal surfaces in Minkowski 3-space. C. R. Math. Acad. Sci. Paris **344**, 785–790 (2007) (pp. 20, 254)
[MRR11] Mazet, L., Rodríguez, M.M., Rosenberg, H.: The Dirichlet problem for the minimal surface equation, with possible infinite boundary data, over domains in a Riemannian surface. Proc. Lond. Math. Soc. **102**, 985–1023 (2011) (pp. 2, 134)
[MP01] Mazzeo, R., Pacard, F.: Constant mean curvature surfaces with Delaunay ends. Commun. Anal. Geom. **9**, 169–237 (2001) (p. 5)
[MP02] Mazzeo, R., Pacard, F.: Bifurcating Nodoids. Topology and Geometry: Commemorating SISTAG. Contemp. Math., pp. 169–186. Am. Math. Soc., Providence (2002) (p. 5)
[MPP01] Mazzeo, R., Pacard, F., Pollack, D.: Connected sums of constant mean curvature surfaces in Euclidean 3 space. J. Reine Angew. Math. **536**, 115–165 (2001) (p. 5)
[MPP05] Mazzeo, R., Pacard, F., Pollack, D.: The conformal theory of Alexandrov embedded constant mean curvature surfaces in $\mathbb{R}^3$. In: Hofmann, D. (ed.) Global Theory of Minimal Surfaces. Clay Mathematics Proceedings, vol. 2, pp. 525–559. Am. Math. Soc., Providence (2005) (p. 5)
[Mcc99] McCuan, J.: Continua of H-graphs: convexity and isoperimetric stability. Calc. Var. Partial Differ. Equ. **9**, 297–325 (1999) (pp. 9, 11, 141, 271)
[Mcc02] McCuan, J.: A generalized height estimate for graphs, Serrin's corner lemma, and applications to a conjecture of Rosenberg. In: Minimal surfaces, Geometric Analysis and Symplectic Geometry, Baltimore, MD, 1999. Adv. Stud. Pure Math., vol. 34, pp. 201–217. Math. Soc. Japan, Tokyo (2002) (pp. 13, 67)
[Mee78] Meeks, W.H. III: Lectures on Plateau's Problem. Inst. Mat. Pura Apl., Rio de Janeiro (1978) (pp. 11, 65)
[Mee88] Meeks, W.H. III: The topology and geometry of embedded surfaces of constant mean curvature. J. Differ. Geom. **27**, 539–552 (1988) (pp. 5, 17, 1, 4, 29, 71, 173, 176)
[MP12] Meeks, W.H. III, Pérez, J.: Constant mean curvature surfaces in metric Lie groups. Geom. Anal.: Part. Differ. Equ. Surf. Contemp. Math. **570**, 25–110 (2012) (p. x)
[MY82a] Meeks, W.H. III, Yau, S.T.: The classical Plateau problem and the topology of three dimensional manifolds. Topology **21**, 409–442 (1982) (p. 11)
[MY82b] Meeks, W.H. III, Yau, S.T.: The existence of embedded minimal surfaces and the problem of uniqueness. Math. Z. **179**, 151–168 (1982) (p. 11)
[Meu85] Meusnier, M.: Mémoire sur la courbure des surfaces. Mémoires de Mathematique et de Physique (de savans étrangers) de l'Academie **X**, 447–550 (1785) (p. 3)
[Mik80] Miklyukov, V.M.: On a new approach to Bernstein's theorem and related questions for equations of minimal surface type. Mat. Sb. **108**, 268–289 (1979). English transl. in Math. USSR **36**, 251–271 (1980) (pp. 14, 186)

[Mon01] Montiel, S.: A height estimate for H-surfaces and existence of H-graphs. Am. J. Math. **123**, 505–514 (2001) (pp. 20, 152)
[MR85] Montiel, S., Ros, A.: Minimal immersions of surfaces by the first eigenfunctions and conformal area. Invent. Math. **83**, 153–166 (1985) (p. 6)
[MR91] Montiel, S., Ros, A.: Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures. In: Pitman Monogr. Surveys Pure Appl. Math., vol. 52, pp. 279–296. Longman Sci. Tech, Harlow (1991) (p. 4)
[MR09] Montiel, S., Ros, A.: Curves and Surfaces. Graduate Studies in Mathematics, vol. 69. Am. Math. Soc., Providence (2009) (p. viii)
[Mor95] Morgan, F.: Geometric Measure Theory: A Beginner's Guide, 2nd edn. Academic Press, San Diego (1995) (pp. 8, 20)
[Mul02] Müller, F.: Analyticity of solutions for semilinear elliptic systems of second order. Calc. Var. Partial Differ. Equ. **15**, 257–288 (2002) (pp. 18, 30)
[NE96] Nelli, B., Earp, R.: Some properties of hypersurfaces of prescribed mean curvature in $\mathbb{H}^{n+1}$. Bull. Sci. Math. **120**, 537–553 (1996) (pp. 10, 198)
[NEST08] Nelli, B., Earp, R., Santos, W., Toubiana, E.: Uniqueness of H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$, $|H| \leq 1/2$, with boundary one or two parallel horizontal circles. Ann. Glob. Anal. Geom. **33**, 307–321 (2008) (p. x)
[NR95] Nelli, B., Rosenberg, H.: Some remarks on embedded hypersurfaces in hyperbolic space of constant mean curvature and spherical boundary. Ann. Glob. Anal. Geom. **13**, 23–30 (1995) (pp. 10, 11, 17, 198, 199, 205)
[NeSe99] Nelli, B., Semmler, B.: Some remarks on compact constant mean curvature hypersurfaces in a halfspace of $\mathbb{H}^{n+1}$. J. Geom. **64**, 128–140 (1999) (pp. 16, 118)
[NeSp96] Nelli, B., Spruck, J.: On the existence and uniqueness of constant mean curvature hypersurfaces in hyperbolic space. In: Jost, J. (ed.) Geometric Analysis and the Calculus of Variations, pp. 253–266. International Press, Cambridge (1996) (pp. 2, 220)
[Nit64] Nitsche, J.C.C.: A supplement to the condition of J. Douglas. Rend. Circ. Mat. Palermo **13**, 192–198 (1964) (pp. 12, 66)
[Nit65] Nitsche, J.C.C.: On new results in the theory of minimal surfaces. Bull., New Ser., Am. Math. Soc. **71**, 195–270 (1965) (pp. 11, 183)
[Nit89] Nitsche, J.C.C.: Lectures on Minimal Surfaces, vol. 1. Cambridge University Press, Cambridge (1989) (pp. vii, x, 3, 8, 12, 66)
[Nit02] Nitsche, P.A.: Existence of prescribed mean curvature graphs in hyperbolic space. Manuscr. Math. **108**, 349–367 (2002) (pp. 1, 219)
[Nor05] Norbury, P.: The 12th problem. Aust. Math. Soc. Gaz. **32**, 244–246 (2005) (p. 6)
[One83] O'Neill, B.: SemiRiemannian Geometry with Application to General Relativity. Academic Press, New York (1983) (pp. 2, 236)
[Opr00] Oprea, J.: The Mathematics of Soap Films: Explorations with Maple. Student Mathematical Library, vol. 10. Am. Math. Soc., Providence (2000) (p. vii)
[Oss86] Osserman, R.: A Survey of Minimal Surfaces. Dover, New York (1986) (pp. vii, x, 3, 14, 50)
[PR10] Pacard, F., Rosenberg, H.: Attaching handles to Delaunay nodoids (2010). arXiv:1010.4974 (p. 5)
[Pal00] Palmer, B.: Uniqueness theorems for Willmore surfaces with fixed and free boundaries. Indiana Univ. Math. J. **49**, 1581–1601 (2000) (pp. 8, 130)
[Par05] Park, S.: Every ring type spanner in a wedge is spherical. Math. Ann. **332**, 475–482 (2005) (pp. 13, 67)
[PP79] Payne, L.E., Philippin, G.A.: Some maximum principles for nonlinear elliptic equations in divergence form with applications to capillary surfaces and to surfaces of constant mean curvature. Nonlinear Anal. Th. Meth. Appl. **3**, 193–211 (1979) (pp. 19, 151)
[PSS77] Payne, L.E., Sperb, R.P., Stakgold, I.: On Hopf type maximum principles for convex domains. Nonlinear Anal. **1**, 547–559 (1977) (pp. 4, 106)

[Per10] Perdomo, O.: Embedded constant mean curvature hypersurfaces on spheres. Asian J. Math. **14**, 73–108 (2010) (p. 7)
[Pin09] Pinheiro, A.L.: A Jenkins-Serrin theorem in $M^2 \times \mathbb{R}$. Bull. Braz. Math. Soc. **40**, 117–148 (2009) (pp. x, 2, 134)
[PS89] Pinkall, U., Sterling, I.: On the classification of constant mean curvature tori. Ann. Math. **130**, 407–451 (1989) (pp. 5, 6)
[Pla73] Plateau, J.A.F.: Statique expérimentale et théoretique des liquides soumis aux seule forces moléculaires, vols. I, II. Gautier-Villars, Paris (1873) (p. 7)
[Poi85] Poincaré, H.: Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation. Acta Math. **7**, 259–380 (1885) (pp. 7, 61)
[PW67] Protter, M.H., Weinberger, H.: Maximum Principles in Differential Equations. Prentice-Hall, Englewood Cliffs (1967) (pp. 8, 44)
[Pyo10] Pyo, J.: Minimal annuli with constant contact angle along the planar boundaries. Geom. Dedic. **146**, 159–164 (2010) (pp. 13, 67)
[Qui85] Quien, N.: Plateau's problem in Minkowski space. Analysis (Munich) **5**, 43–60 (1985) (pp. 21, 255)
[Rad30a] Radó, T.: The problem of least area and the problem of Plateau. Math. Z. **32**, 763–795 (1930) (p. 8)
[Rad30b] Radó, T.: On Plateau's problem. Ann. Math. **31**, 457–469 (1930) (p. 8)
[Rad32] Radó, T.: Contributions to the theory of minimal surfaces. Acta Litt. Sci. Univ. Szeged **2**, 1–20 (1932–1934) (pp. 8, 11, 31, 163)
[Rad51] Radó, T.: On the Problem of Plateau. Chelsea, New York (1951) (pp. 8, 11, 31, 163)
[Rat01] Ratzkin, J.: An end-to-end gluing construction for surfaces of constant mean curvature. Ph. Doctoral thesis, University of Washington (2001) (p. 5)
[Rei77] Reilly, R.: Applications of the Hessian operator in a Riemannian manifold. Indiana Univ. Math. J. **26**, 459–472 (1977) (p. 4)
[Rei82] Reilly, R.: Mean curvature, the Laplacian, and soap bubbles. Am. Math. Mon. **89**, 180–188 (1982) and 197–198 (pp. 20, 32)
[Rie98] Riemann, B.: Oeuvres mathématiques, pp. 341–347. Gauthiers-Villars, Paris (1898) (pp. 11, 65)
[Rip01] Ripoll, J.: Some characterization, uniqueness and existence results for Euclidean graphs of constant mean curvature with planar boundary. Pac. J. Math. **198**, 175–196 (2001) (pp. 1, 13, 30, 133, 145, 162)
[Rip02] Ripoll, J.: Some existence results and gradient estimates of solutions of the Dirichlet problem for the constant mean curvature equation in convex domains. J. Differ. Equ. **181**, 230–241 (2002) (pp. 13, 23, 145, 155)
[RS08] Ripoll, J., Sauer, L.: A note on the Dirichlet problem for the minimal surface equation in nonconvex planar domains. Mat. Contemp. **35**, 177–183 (2008) (pp. 30, 162)
[Ros87] Ros, A.: Compact hypersurfaces with constant higher order mean curvatures. Rev. Mat. Iberoam. **3**, 447–453 (1987) (p. 4)
[Ros88] Ros, A.: Compact hypersurfaces with constant scalar curvature and a congruence theorem. J. Differ. Geom. **27**, 215–220 (1988) (p. 4)
[Ros95] Ros, A.: A two-piece property for compact minimal surfaces in a three-sphere. Indiana Univ. Math. J. **44**, 841–849 (1995) (p. 6)
[RR96] Ros, A., Rosenberg, H.: Constant mean curvature surfaces in a half-space of $\mathbb{R}^3$ with boundary in the boundary of the half-space. J. Differ. Geom. **44**, 807–817 (1996) (pp. 1, 103)
[RR10] Ros, A., Rosenberg, H.: Properly embedded surfaces with constant mean curvature. Am. J. Math. **132**, 1429–1443 (2010) (pp. 1, 4, 173, 176)
[Rsb93] Rosenberg, H.: Hypersurfaces of constant curvature in space forms. Bull. Sci. Math. **117**, 211–239 (1993) (pp. 16, 19, 28, 31)
[Rss68] Ross, D.K.: The shape and energy of a revolving liquid mass held together by surface tension. Aust. J. Phys. **21**, 823–835 (1968) (pp. 7, 61)

[Rsm98] Rossman, W.: Minimal surfaces with planar boundary curves. Kyushu J. Math. **52**, 209–225 (1998) (pp. 12, 66)
[Ruc79] Ruchert, H.: Ein eindeutigkeitssatz für flaächen konstanter mittlerer krümmung. Arch. Math. **33**, 91–104 (1979) (pp. 9, 131)
[Sak89] Sakaguchi, S.: Uniqueness of critical point of the solution to the prescribed constant mean curvature equation over convex domain in $\mathbb{R}^2$. Recent topics in nonlinear PDE. In: North-Holland Math. Stud., vol. 160, Kyoto, 1988, pp. 129–151. North-Holland, Amsterdam (1989) (pp. 11, 271)
[Sal89] Salavessa, I.M.C.: Graphs with parallel mean curvature. Proc. Am. Math. Soc. **107**, 449–458 (1989) (pp. 2, 220)
[Sau06] Sauvigny, F.: Foundations and Integral Representation, Vol. 1. Functional Analytic Methods, Vol. 2. Partial Differential Equations, Springer, Berlin (2006) (pp. 1, 133)
[Sch83] Schoen, R.: Uniqueness, symmetry, and embeddedness of minimal surfaces. J. Differ. Geom. **18**, 791–809 (1983) (pp. 2, 5, 11, 14, 16, 56, 59, 65, 68, 70)
[Sch88] Schoen, R.: The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation. Commun. Pure Appl. Math. **41**, 317–392 (1988) (p. 5)
[Swa04] Schwab, D.: Hypersurfaces of prescribed mean curvature in central projection. I. Arch. Math. **82**, 245–262 (2004) (pp. 33, 165)
[Swa05] Schwab, D.: Hypersurfaces of prescribed mean curvature in central projection. II. Arch. Math. **84**, 171–182 (2005) (pp. 33, 165)
[Swz84] Schwarz, H.A.: Beweis des Satzes, dass die Kugel kleinere Oberfläche besitzt als jeder andere Körper gleichen Volumens. Nachrichten der Königlichen Gesellschaft für Wissenschaften Göttingen, pp. 1–13 (1884) (pp. 2, 12, 114)
[Sem00] Semmler, B.: The topology of large H-surfaces bounded by a convex curve. Ann. Sci. Éc. Norm. Super. **33**, 345–359 (2000) (pp. 16, 118)
[Sen11] Senni, C.: Non existence of constant mean curvature graphs on circular annuli of $\mathbb{H}^2$ (2011). arXiv:1011.6583 (p. x)
[Ser69a] Serrin, J.: On surfaces of constant mean curvature which span a given space curve. Math. Z. **112**, 77–88 (1969) (pp. 17, 15, 9, 11, 29, 51, 71, 89, 91)
[Ser69b] Serrin, J.: The problem of Dirichlet for quasilinear elliptic equations with many independent variables. Philos. Trans. R. Soc. Lond. Ser. A **264**, 413–496 (1969) (pp. 11, 20, 21, 23, 34, 152, 153, 155, 166)
[Ser70] Serrin, J.: The Dirichlet problem for surfaces of constant mean curvature. Proc. Lond. Math. Soc. **21**, 361–384 (1970) (pp. 20, 7, 152, 179)
[Ser71] Serrin, J.: A symmetry problem in potential theory. Arch. Ration. Mech. Anal. **43**, 304–318 (1971) (pp. 13, 14, 67, 68)
[Shi56] Shiffman, M.: On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes. Ann. Math. **63**, 77–90 (1956) (pp. 11, 65)
[SS09] Silva, D., Spruck, J.: Rearrangements and radial graphs of constant mean curvature in hyperbolic space. Calc. Var. Partial Differ. Equ. **34**, 73–95 (2009) (pp. 1, 219)
[Sim97] Simon, L.: The minimal surface equation. In: Osserman, R. (ed.) Geometry V, Minimal Surfaces. Encycl. Math. Sci., vol. 90, pp. 239–266. Springer, Berlin (1997) (pp. 1, 133)
[Spe81] Sperb, R.P.: Maximum Principles and Their Applications. Academic Press, New York (1981) (pp. 4, 106)
[ST79] Sperb, R.P., Stakgold, I.: Estimates for membranes of varying density. Appl. Ann. **8**, 301–318 (1979) (pp. 4, 106)
[Spi79] Spivak, M.: A Comprehensive Introduction to Differential Geometry. Publish Or Perish, Houston (1979) (pp. 4, 6, 5, 60, 265)
[Spr02] Spruck, J.: Two-dimensional minimal graphs over unbounded domains. J. Inst. Math. Jussieu **1**, 631–640 (2002) (pp. 1, 173)
[Spr07] Spruck, J.: Interior gradient estimates and existence theorems for constant mean curvature graphs in $M \times R$. Pure Appl. Math. Q. **3**, 785–800 (2007) (p. x)

[Ste76] Steffen, K.: On the existence of surfaces with prescribed mean curvature and boundary. Math. Z. **146**, 113–135 (1976) (pp. 8, 20, 152)
[Ste86] Steffen, K.: On the nonuniqueness of surfaces with constant mean curvature spanning a given contour. Arch. Ration. Mech. Anal. **94**, 101–122 (1986) (p. 9)
[Ste99] Steffen, K.: Parametric surfaces of prescribed mean curvature. In: Hildebrandt, S., Struwe, M. (eds.) Calculus of Variations and Geometric Evolution Problems, Lectures Given at the 2nd Session of the Centro Internazionale Matematico Estivo (CIME), Cetraro, Italy, June 15–22, 1996. Lecture Notes in Mathematics, vol. 1713, pp. 211–265. Springer, Berlin (1999) (p. 8)
[SW78] Steffen, K., Wente, H.C.: The non-existence of branch points in solutions to certain classes of Plateau type variational problems. Math. Z. **163**, 211–238 (1978) (p. 8)
[Stn42] Steiner, J.: Sur le maximum et le minimum des figures dans le plan, sur la sphére et dans l'espace en général. J. Reine Angew. Math. **24**, 93–152 (1842) (pp. 2, 12, 114)
[Str85] Struwe, M.: Large H-surfaces via the mountain-pass-lemma. Math. Ann. **270**, 441–459 (1985) (p. 9)
[Str86] Struwe, M.: Nonuniqueness in the Plateau problem for surfaces of constant mean curvature. Arch. Ration. Mech. Anal. **93**, 135–157 (1986) (p. 9)
[Str88] Struwe, M.: Plateau's Problem and the Calculus of Variations. Mathematical Notes. Princeton Univ. Press, Princeton (1988) (pp. x, 8)
[Str00] Struwe, M.: Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer, Berlin (2000) (pp. x, 8)
[Tau81] Tausch, E.: The n-dimensional least area problem for boundaries on a convex cone. Arch. Ration. Mech. Anal. **75**, 407–416 (1981) (pp. 31, 163)
[Tay76] Taylor, J.E.: The structure of singularities in solutions to ellipsoidad problems with constraints in $\mathbb{R}^3$. Ann. Math. **103**, 541–546 (1976) (pp. 12, 114)
[Tho12] Thorpe, B.S.: The maximal graph Dirichlet problem in semi-Euclidean spaces. Commun. Anal. Geom. **20**, 255–270 (2012) (pp. 21, 255)
[Thu97] Thurston, W.: Three-Dimensional Geometry and Topology. Princeton Math. Ser., vol. 35. Princeton Univ. Press, Princeton (1997) (p. x)
[Tom75] Tomi, F.: A perturbation theorem for surfaces of constant mean curvature. Math. Z. **141**, 253–264 (1975) (pp. 23, 5, 35, 137)
[Ton96] Tonegawa, Y.: Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space. Math. Z. **221**, 591–615 (1996) (pp. 2, 220)
[Tre82] Treibergs, A.: Entire spacelike hypersurfaces of constant mean curvature in Minkowski space. Invent. Math. **66**, 39–56 (1982) (pp. 21, 255)
[UY92] Umehara, M., Yamada, K.: A deformation of tori with constant mean curvature in $\mathbb{R}^3$ to those in other space forms. Trans. Am. Math. Soc. **330**, 845–857 (1992) (p. 5)
[Urb90] Urbano, F.: Minimal surfaces with low index in the three-dimensional sphere. Proc. Am. Math. Soc. **108**, 989–992 (1990) (p. 6)
[Urs03] Urbas, J.: The Dirichlet problem for the equation of prescribed scalar curvature in Minkowski space. Calc. Var. Partial Differ. Equ. **18**, 307–316 (2003) (pp. 21, 255)
[Wal89] Walter, R.: Constant mean curvature tori with spherical curvature lines in non-Euclidean geometry. Manuscr. Math. **63**, 343–363 (1989) (p. 5)
[Wan90] Wang, A.N.: Constant mean curvature surfaces on a strip. Pac. J. Math. **145**, 395–396 (1990) (pp. 10, 182)
[Wng12] Wang, X.-J.: Counterexample to the convexity of level sets of solutions to the mean curvature equation. Preprint (2012) (pp. 11, 271)
[Wei96] Weinstein, T.: An Introduction to Lorentz Surfaces. Walter de Gruyter, Berlin (1996) (pp. 2, 236)
[Wen69] Wente, H.C.: An existence theorem for surfaces of constant mean curvature. J. Math. Anal. Appl. **26**, 318–344 (1969) (pp. 8, 20, 152)
[Wen71] Wente, H.C.: A general existence theorem for surfaces of constant mean curvature. Math. Z. **120**, 277–288 (1971) (pp. 8, 9)

[Wen80a] Wente, H.C.: The symmetry of sessile and pendant drops. Pac. J. Math. **88**, 387–397 (1980) (pp. 9, 63)
[Wen80b] Wente, H.C.: Large solutions to the volume constrained Plateau problem. Arch. Ration. Mech. Anal. **75**, 59–77 (1980) (pp. 8, 4, 106)
[Wen82] Wente, H.C.: The symmetry of rotating fluid bodies. Manuscr. Math. **39**, 287–296 (1982) (pp. 7, 61)
[Wen86] Wente, H.C.: Counterexample to a conjecture of H. Hopf. Pac. J. Math. **121**, 193–243 (1986) (pp. x, 4)
[Wen91] Wente, H.C.: A note on the stability theorem of J.L. Barbosa and M. do Carmo for closed surfaces of constant mean curvature. Pac. J. Math. **147**, 375–379 (1991) (pp. 23, 35)
[Wer57] Werner, H.: Problem von Douglas für Flächen konstanter mittlerer Krümmung. Math. Ann. **135**, 303–319 (1957) (p. 8)
[Whi87] White, B.: Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals. Invent. Math. **88**, 243–256 (1987) (pp. 14, 116)
[You05] Young, T.: An essay of the cohesion of fluids. Philos. Trans. R. Soc. London **95**, 65–87 (1805) (p. 3)
[Zha05] Zhang, Z.: A remark on the mean curvature of a graph-like hypersurface in hyperbolic space. J. Math. Anal. Appl. **305**, 491–501 (2005) (pp. 1, 219)

Index

A
A priori estimates, 137
Abresch-Rosenberg theorem, x
Alexander duality, 42, 70
Alexandrov reflection method, 4, 56
  in $\mathbb{H}^3$, 200
Alexandrov theorem, 4, 60
Algebraic area of a curve, 86
Algebraic volume, 14
Angular momentum, 84
Archimedes theorem, 107
Area, 14
Area element, 14
Area estimate by Alías-Palmer, 131
Area estimate by Finn, 112
Area functional, 15
Area of a spherical cap, 109
Ascoli-Arzelá theorem, 137

B
Balancing formula, 84
Barbosa theorem, 92, 94
Barbosa-do Carmo isoperimetric inequality, 108, 123, 215
Barbosa-do Carmo theorem, 4, 35
Barrier surfaces, 145
Bartnik-Simon theorem, 255
Bernstein theorem, 50, 173
  in $\mathbb{L}^3$, 254
Boundary of an immersion, 13
Bounded slope condition, 142, 155
Boundedness of an H-surface, 71
Brito-Earp theorem, 89, 200
Brito-Earp-Meeks-Rosenberg conjecture, 100
Brito-Earp-Meeks-Rosenberg theorem, 97

C
Capillary equation for rotational surfaces, 63
Capillary surface, 12
Catenoid, 43, 65, 119
Cauchy-Schwarz inequality for timelike vectors, 236
Circle in $\mathbb{L}^3$, 243
Cmc surface, vii, 17
  spanning two coaxial circles, 64
Coarea formula, 104, 252
Codazzi equations for an H-surface, 25
Collin example, 182
Comparison principle, 37, 39
  in $\mathbb{H}^3$, 194
  in $\mathbb{L}^3$, 242
Conformal coordinates, 24, 128
Constant mean curvature equation, 11, 38
Constant mean curvature surface, 17
Continuity method, 134
  for radial graphs, 166
  in $\mathbb{H}^3$, 224
  in $\mathbb{L}^3$, 254
Convex hull property
  in $\mathbb{H}^3$, 195
  in $\mathbb{R}^3$, 49

D
Delaunay surfaces, 5, 157
  in $\mathbb{L}^3$, 256
Diameter, 106, 249
Dirichlet integral, 8
Dirichlet problem, 10, 134
  for radial graphs, 163
  in $\mathbb{H}^3$, 219
  in $\mathbb{L}^3$, 253
  on bounded convex domains, 145
  on unbounded convex domains, 179

R. López, *Constant Mean Curvature Surfaces with Boundary*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-39626-7,

© Springer-Verlag Berlin Heidelberg 2013

Dirichlet problem (*cont.*)
on unbounded domains, 173
uniqueness, 48
uniqueness for radial graphs, 164
DPW method, x, 5

E
Embedding, 13
Equidistant surface, 192, 220

F
Faber-Krahn inequality, 131
Finn theorem, 11
First eigenvalue of a hemisphere, 23
First variation for the area, 16, 261
First variation for the volume, 18, 263
Flux formula, 82
for a minimal surface, 85
in $\mathbb{H}^3$, 199
in $\mathbb{L}^3$, 246
Flux of the surface, 84
Foliation of $\mathbb{H}^3$ by umbilical surfaces, 196
Fredholm operator, 137
Future orientation, 237

G
Gauss curvature in $\mathbb{L}^3$, 240
Gauss formula, 26
in $\mathbb{L}^3$, 240
Gauss-Bonnet formula, 62
Geodesic cylinder in $\mathbb{H}^3$, 207
Geodesic graph in $\mathbb{H}^3$, 205, 219
Geodesic plane, 192
Go null, 255
Gradient estimates, 139, 145, 147
for radial graphs, 169
in $\mathbb{H}^3$, 226, 230
in $\mathbb{L}^3$, 254
Graph on a geodesic plane, 228
Graph on a horosphere, 223
Graph on an equidistant surface, 228

H
H-graph, 134
H-small surface, 93
H-surface, 17
H-surfaces
in balls, 52
in cylinders, 52
in slabs, 53
with boundary in a sphere, 72
with positive Gauss curvature, 62
Half catenoid, 60
Half cylinder, 53
Harnack's principle, 181
Hartman condition, 143
Height estimate, 29, 175
for an embedded cmc surface, 71
for radial graphs, 166
in $\mathbb{H}^3$, 222, 225, 228
in $\mathbb{L}^3$, 249, 250
in terms of the area, 104
Heinz estimate, 9, 87
Hemisphere, 9
Hessian, 38
Hildebrandt solutions, 93
Hopf conjecture, 4
Hopf differential, 25
Hopf maximum principle, 45
boundary version, 45
corner point, 67
interior version, 45
Hopf theorem, 4, 26
Horosphere, 192, 220
Hyperbolic cap in $\mathbb{L}^3$, 239, 243
Hyperbolic plane, 239
Hyperbolic reflection, 193
Hyperbolic rotation, 193
Hyperbolic space
Lorentz-Minkowski model, 217
upper half-space model, 190
Hyperbolic sphere, 193, 220
Hyperbolic translation, 193

I
Immersion with boundary, 13
Interfaces, 3
Invertibility of the Jacobi operator, 136
Isoperimetric problem, vii, 2, 15, 115
Isoperimetric region, 114

J
Jacobi operator, 22, 135
in $\mathbb{H}^3$, 217, 228
in $\mathbb{L}^3$, 254
Jellet theorem, 3, 31, 166
Jenkins-Serrin theorem, 183
Joachimstal theorem, 85, 88

K
Kapouleas examples, 10, 55, 71
Killing cylinder in $\mathbb{H}^3$, 207
Killing graph in $\mathbb{H}^3$, 205
Killing vector field, 83
in $\mathbb{H}^3$, 199
Koiso theorem, 42
in $\mathbb{H}^3$, 204

L
Laplace equation, 3, 61
Laplacian of the Gauss map, 27
in $\mathbb{H}^3$, 29
Laplacian of the support function, 27
Laplacian of the vector position, 27
in $\mathbb{H}^3$, 29
Laplacian-Beltrami, 21
Large solution, 9, 121
Large spherical cap, 9
Large volume, 120
Liebmann theorem, 4, 62
Light-cone, 236
Lightlike vector, 236
Linear elliptic equation, 44
Linear elliptic operator, 44
Linear momentum, 83
Liouville theorem, 26
Liquid drop, 3, 63
Lorentz-Minkowski space, 236

M
Maximal surface, 242
Maximum principle, 48
Mean convex side, 53, 197
Mean curvature, 16
by conformal metrics, 190
in $\mathbb{H}^3$, 191
in $\mathbb{L}^3$, 240
Mean curvature equation
for radial graphs, 163
in $\mathbb{H}^3$, 193, 223
in divergence form, 10, 38
in $\mathbb{L}^3$, 242
Meeks' conjecture, 65
Method of moving planes, 56
Minimal surface, 2, 17
variational characterization, 17
Minkowski formula, 113, 120
Modulus of continuity, 181, 185
Monotonicity for H-graphs
in $\mathbb{H}^3$, 223
in $\mathbb{R}^3$, 41
Monotonicity formula, 103, 106
in $\mathbb{L}^3$, 251

N
Nelli-Rosenberg theorem, 199
Nodal domain, 130
Nodal line, 104, 252
Nodoid, 64, 156
Noether theorem, 83
Non parametric case, 10

O
Overdetermined problem, 67

P
Parabolic graph in $\mathbb{H}^3$, 210
Parabolic translation, 193
Payne-Philippin theorem, 151
Perron method, 179
Physical equilibrium, 3
Plateau problem, 7, 8
Principal curvatures, 16

Q
Quarter of cylinder, 148

R
r-Mean curvature, 31
Radial graph, 162
Radó theorem, 11, 163
Rayleigh characterization, 22, 23
Regularity of solutions, 134, 145
Rellich's conjecture, 9
Riemann minimal examples, 65
Rolling property, 94, 145
Rosenberg conjecture, 270
Rossman theorem, 66
Rotating liquid drops, 61

S
Scherk surface, 51, 183
Second fundamental form, 16
Second variation for the area, 21, 265
Serrin corner lemma, 67
Serrin estimate, 29, 225, 229
Serrin theorem, 11, 152
Shiffman theorem, 65
Slab, 53, 95
Slope of a graph, 139
Small spherical cap, 9
Small surface, 92
Small volume, 110
Soap bubbles, 3
Soap films, 3, 85
Sobolev space, 22
Solvability of the minimal surface equation, 144
Spacelike
circle, 243
cylinder, 251
immersion, 236
orientability, 237
plane, 239
surface, 236
vector, 236
Sphere, 3

Spherical cap, 9
area, 124
in $\mathbb{H}^3$, 197
area, 215
Stability, 21
Stability of a sphere, 22
Stability of a spherical cap, 22
Star-shaped surface, 31
Strong stability
of a cmc graph, 34
of a spherical cap, 22, 128
Strongly stability, 21
Subfunction, 180, 185, 186
Subsolution, 180
Surface with constant Gauss curvature, 31
Surface with constant mean curvature, 17
Sym-Bobenko formula, x

T
Tangency principle, 44, 47
in $\mathbb{H}^3$, 194
in $\mathbb{L}^3$, 242
Timelike vector, 236

U
Umbilical points in a cmc surface, 26
Umbilical surfaces
in $\mathbb{H}^3$
spanning a circle, 198
in $\mathbb{H}^3$, 192, 220
in $\mathbb{L}^3$, 239
Unduloid, 178
Uniform area estimates for a cmc graph, 116
Uniform exterior circle condition, 159, 183
Uniform interior sphere condition, 45
Uniformly elliptic operator, 45
Uniqueness of the Dirichlet problem, 182, 186
for Killing graphs of $\mathbb{H}^3$, 206
in $\mathbb{L}^3$, 242
Uniqueness on right cylinders, 89

V
Variation of an immersion, 15, 261
admissible, 15
existence, 18
volume preserving, 18
Variational characterization of a constant mean curvature surface, 19
Variational vector field, 15, 261
Volume
of a closed surface, 14
of a graph, 15
Volume functional, 15, 263

W
Wang example, 182
Weingarten endomorphism, 16
in $\mathbb{L}^3$, 240
Weingarten formula, 26
Wente theorem, 152
Wente torus, x, 4